Chapter 4

SELECTION OF A BETTER COMPONENT IN BIVARIATE LIFETIME MODELS

4.1 Introduction:

Stochastic orders are powerful tool of comparing two random variables or two distributions. In the present chapter we apply various orderings to select better component between the two components of a parallel system. Suppose we have a two unit parallel system having dependent lifetime distributions of the units. Suppose C_1 and C_2 are the units of parallel system and T_1 and T_2 are random variables denoting their respective marginal lifetimes. We assume that $F\,(t_1, t_2,\ \underline{\theta}\,)$ be the joint distribution of $(T_1, T_2)'$ and $F_1(t, \underline{\theta})$, $F_2(t, \underline{\theta})$ be their respective marginal cumulative distribution functions. We assume that $\underline{\theta}$ is the vector of unknown parameter having dimension p , $p \geq 2$ and marginal distributions of T_1 and T_2 have mean θ_1 and θ_2 respectively. Where θ_1 and θ_2 can be real valued functions of components of $\underline{\theta}$. The problem is to select better component between C_1 and C_2.

Selection of better component for bivariate exponential distribution has been studied by Hyakutake (1992) and Hanagal (1997). In both the papers the criterion of betterness is studied with respect to the mean life of components. But we know that comparison of two random variables by some other criterion is more appealing. In the present study we give some other criterion and compare these criteria through the probability of correct selection for some bivariate models.

Section 4.2 of this chapter is devoted to the selection of better component through mean and stochastic orders. In section 4.3 we discuss procedure of selection of better component by counts. In section 4.4 we discuss selection procedure based on sample mean. In Section 4.5 we discuss the selection of better component by procedure based on maximum likelihood estimators. Asymptotic relative efficiency and example of Block-Basu's BVE Model are given in the last section.

4.2 Procedure based on mean and stochastic orders:

Betterness of component C_1 with respect to C_2 can be defined by number of ways. The some possible ways are as follows.

The component C_1 is said to be better than component C_2 if

(i) $P_{\underline{\theta}}(T_1 < T_2) \not< P_{\underline{\theta}}(T_1 \geq T_2)$.

(ii) E(T_1) > E(T_2), where E(.) is the expectation of random variables.

(iii) $T_1 \geq_{st} T_2$. That is if $F_1(t_1, \underline{\theta}) \leq F_2(t_2, \underline{\theta})$, where $\geq_{st}$ stands for stochastically greater.

(iv) $T_1 \geq_{hr} T_2$. That is if $r(t_1, \underline{\theta}) \leq q(t_2, \underline{\theta})$, where $\geq_{hr}$ stands for greater in hazard rate order and $r(\cdot, \underline{\theta})$ and $q(\cdot, \underline{\theta})$ are hazard rate functions of T_1 and T_2 respectively.

(v) $T_1 \geq_{lr} T_2$. That is if $\frac{f_1(t_1, \underline{\theta})}{f_2(t_2, \underline{\theta})}$ increases in t, where $\geq_{lr}$ stands for greater in likelihood ratio order and $f_1(\cdot, \underline{\theta})$ and $f_2(\cdot, \underline{\theta})$ are the density functions of T_1 and T_2 respectively.

(vi) $T_1 \geq_{mrl} T_2$. That is if $m(t_1, \underline{\theta}) \geq l(t_2, \underline{\theta})$, where $\geq_{mrl}$ stands for greater in mean residual life order and $m(\cdot, \underline{\theta})$ and $l(\cdot, \underline{\theta})$ are the mean residual functions of the T_1 and T_2 respectively.

(vii) $T_1 \geq_{disp} T_2$. That is if ,

$$F_1^{-1}(t_2, \theta_2) - F_1^{-1}(t_1, \theta_1) \geq F_2^{-1}(t_2, \theta_2) - F_2^{-1}(t_1, \theta_1),$$

for $0 < \theta_1 \leq \theta_2 < 1$, where $\geq_{disp}$ stands for greater in dispersive order and $F_i^{-1}(\cdot, \underline{\theta})$ is inverse c.d.f. of T_i , $i = 1,2$.

Above criteria can be used to select a better component between the two. In the following we discuss some procedures to select a better component. These procedures are extensions of work due to Hyakutake (1992) and Hanagal (1997).

4.3 Procedure based on counts:

Let (T_{1i}, T_{2i}), $i = 1,2,\ldots,n$ be a random sample of size n from $F(t_1, t_2; \theta)$. Suppose n_1 (n_2) be the number of observations such that $T_1 < T_2$, $(T_1 > T_2)$ and n_3 be number of observations such that $T_1 = T_2$.

Let $P_1(\underline{\theta}) = P(T_1 < T_2)$ and $P_2(\underline{\theta}) = P(T_1 > T_2)$.

Hence

$P_3(\underline{\theta}) = 1 - P_1(\theta) - P_2(\theta)$ will be $P(T_1 = T_2)$.

Rule R₁: A component C_1 is said to be better than component C_2 if

$$\frac{n_1}{n} < \frac{n_2}{n}.$$

That is $\left(\frac{n_2 - n_1}{n}\right) > 0$.

We note that (n_1, n_2) follow Trinomial distribution with n, $P_1(\underline{\theta})$, $P_2(\underline{\theta})$.

Therefore

$$P(CS/R_1) = P\left(\frac{n_2 - n_1}{n} > 0\right)$$

$$= P\left[\frac{\sqrt{n}\left(\frac{n_2 - n_1}{n} - \left(P_2(\underline{\theta}) - P_1(\underline{\theta})\right)\right)}{\sigma_1} \geq \frac{-\sqrt{n}\,\mu_1}{\sigma_1}\right]$$

$$= \Phi\left(\sqrt{n}\,\mu_1 / \sigma_1\right),$$

where $\mu_1 = P_2(\theta) - P_1(\theta)$ and $\sigma_1^2 = Var\left(\sqrt{n}\left(\frac{n_2 - n_1}{n}\right)\right)$.

4.4 Procedure based on sample mean:

Rule R_2: Select component C_1 is better than component C_2 if

$$\overline{T_1} = \frac{1}{n}\sum_{i=1}^{n} T_{1i} > \overline{T_2} = \frac{1}{n}\sum_{i=1}^{n} T_{2i} .$$

It is easy to see that since of T_1 and T_2 are dependent and ?

$$P(CS/R_2) = P(\overline{T_1} > \overline{T_2})$$

$$= \Phi(\sqrt{n}\,\mu_2/\sigma_2),$$

where $\mu_2 = E(T_1) - E(T_2)$ and $\sigma_2^2 = Var(\overline{T_1} - \overline{T_2})$.

4.5 Procedure based on maximum likelihood estimators:

Under suitable regularity conditions suppose $\hat{\underline{\theta}}$ be the maximum likelihood estimator (mle) of $\underline{\theta}$. Hence $\hat{\theta}_1$ and $\hat{\theta}_2$ be the mle's of θ_1 and θ_2 respectively.

Rule R_3: Select component C_1 is better than component C_2

If $\hat{\theta}_1 > \hat{\theta}_2$ or $\hat{\theta}_1 - \hat{\theta}_2 > 0$.

We note that $\sqrt{n}\left(\left(\hat{\theta}_1 - \hat{\theta}_2\right) - (\theta_1 - \theta_2)\right) \sim AN(0,\ \sigma_3^2)$.

σ_3^2 can be obtained from the Fisher Information Matrix of order p.

Rule R_4: Select component C_1 is better than component C_2 if for given values of T_1 and T_2,

$$F_{T_1}(t, \underline{\theta}) < F_{T_2}(t, \underline{\theta}), \quad \forall \ t > 0.$$

In parametric set up, $F_{T_1}(t, \hat{\theta})$ and $F_{T_2}(t, \hat{\theta})$ be the respective mle's of $F_{T_1}(t, \underline{\theta})$ and $F_{T_2}(t, \underline{\theta})$ respectively.

Thus component C_1 is better than component C_2 if

$$F_{T_1}(t, \hat{\underline{\theta}}) < F_{T_2}(t, \hat{\underline{\theta}}).$$

It follows that

Then

$$\sqrt{n}\left[\left(F_{T_1}(t_1, \hat{\theta}) - F_{T_2}(t_2, \hat{\theta})\right) - \left(F_{T_1}(t_1, \theta) - F_{T_2}(t_2, \theta)\right)\right]$$

$$\sim \ AN\left(0, \ \sigma^2_{F_{T_1,T_2}}\right),$$

where $\sigma^2_{F_{T_1,T_2}}$ is asymptotic variance of $(F_{T_1}(t, \hat{\underline{\theta}}) - F_{T_2}(t, \hat{\underline{\theta}}))$.

Rule R_5: Based on Hazard rates.

Let $r(t, \underline{\theta})$ and $q(t, \underline{\theta})$ be the hazard rate functions of T_1 and T_2 respectively. Then select component C_1 is better than component C_2 if

$r(t,\hat{\underline{\theta}}) \le q(t,\hat{\underline{\theta}})$ for a given value of t.

Assuming $r(\cdot,\underline{\theta})$ and $q(\cdot,\underline{\theta})$ to be continuous function, $r(t,\hat{\underline{\theta}})$ and $q(t,\hat{\underline{\theta}})$ are consistent and asymptotic normal estimators of $r(\cdot,\underline{\theta})$ and $q(\cdot,\underline{\theta})$ respectively.

Hence

$$\sqrt{n}\left(r(t,\hat{\underline{\theta}})-q(t,\hat{\underline{\theta}})\right) \sim AN\left(r(t,\underline{\theta})-q(t,\underline{\theta}),\ \ \sigma^2_{r,q}(t,\underline{\theta})\right),$$

where $\sigma^2_{r,q}(t,\underline{\theta})$ is asymptotic variance of $\sqrt{n}\left(r(t,\hat{\underline{\theta}})-q(t,\hat{\underline{\theta}})\right)$.

Similarly we can formulate rules related to likelihood ratio order, mean residual life order and dispersive order.

4.6 Asymptotic Relative Efficiency (ARE):

The probability requirement based on the selection procedure R_i, is

$P(CS / R_i) \ge P^*$ where $\frac{1}{2} < P^* < 1$ is fixed constant.

$P(CS / R_i) \ge P^*$ *or* $\Phi(c_i) \ge P^*$ or $m_i \ge \sigma_i^2 Z_p^2 / \mu_i^2,\ i = 1,2,3$,

where $Z_p\ \Phi(c_i) = P^*$.

The minimum sample size required for the i^{th} selection procedure R_i is $m_i = \sigma_i^2 Z_p^2 / \mu_i^2$.

The ARE of the selection procedure R_i with respect to the selection procedure R_j is given by

$$ARE\left(R_i, R_j\right) = \frac{\left(\sigma_j/\mu_j\right)^2}{\left(\sigma_i/\mu_i\right)^2}.$$

A rule R_i is said to be better than R_j if

$$\left(\sigma_i/\mu_i\right)^2 < \left(\sigma_j/\mu_j\right)^2.$$

In the following we discuss rule R_3 for bivariate exponential model due to Block and Basu (1974).

Example 4.1: The random variables X_1 and X_2 follow Absolutely Continuous Bivariate Exponential (ACBVE) distribution having survival function

$$\bar{F}\,(x_1, x_2) = P\left[X_1 > x_1,\, X_2 > x_2\right]$$

$$= \frac{\lambda}{\lambda_1 + \lambda_2} \exp\left[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max\,(x_1, x_2)\right] - \frac{\lambda_3}{\lambda_1 + \lambda_2} \exp\left[(-\lambda \max(x_1, x_2)\right]$$

,

where $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. The probability density function of X_1 and X_2 are given by

$$f\left(x_1, x_2\right) = \frac{\lambda_1\, \lambda\, (\lambda_2 + \lambda_3)}{(\lambda_1 + \lambda_2)} \exp\left[-\lambda_1\, x_1 - (\lambda_2 + \lambda_3)\, x_2\right], \quad x_1 < x_2$$

$$= \frac{\lambda_2 \lambda (\lambda_1 + \lambda_3)}{(\lambda_1 + \lambda_2)} \exp[-\lambda_2 x_2 - (\lambda_1 + \lambda_3) x_1], \quad x_1 \geq x_2 .$$

The marginal probability density function of X_1 and X_2 are given by

$$f_1(x_1) = \frac{\lambda (\lambda_1 + \lambda_3)}{\lambda_1 + \lambda_2} \exp[-(\lambda_1 + \lambda_3) x_1] - \frac{\lambda_3 \lambda}{\lambda_1 + \lambda_2} \exp(-\lambda x_1), \quad x_1 > 0$$

and

$$f_2(x_2) = \frac{\lambda (\lambda_2 + \lambda_3)}{\lambda_1 + \lambda_2} \exp[-(\lambda_2 + \lambda_3) x_2] - \frac{\lambda_3 \lambda}{\lambda_1 + \lambda_2} \exp(-\lambda x_2), \quad x_2 > 0$$

respectively.

The marginal distributions of X_1 (or X_2) is now not exponential but the weighted combination of two exponentials with weights

$$\left[1 + \frac{\lambda_3}{\lambda_1 + \lambda_2}\right] \text{ and } \left[-\frac{\lambda_3}{\lambda_1 + \lambda_2}\right].$$

By taking likelihood function we obtain the Fisher information matrix $I(\lambda_1, \lambda_2, \lambda_3)$ has elements given by Hanagal and Kale (1991).

$$I_{11} = \frac{1}{\lambda^2} - \frac{1}{(\lambda_1 + \lambda_2)^2} + \frac{1}{\lambda_1 (\lambda_1 + \lambda_2)} + \frac{\lambda_2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)^2},$$

$$I_{22} = \frac{1}{\lambda^2} - \frac{1}{(\lambda_1 + \lambda_2)^2} + \frac{1}{\lambda_2 (\lambda_1 + \lambda_2)} + \frac{\lambda_1}{(\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)^2},$$

$$I_{33} = \frac{1}{\lambda^2} + \frac{1}{(\lambda_1 + \lambda_2)}\left[\frac{\lambda_1}{(\lambda_2 + \lambda_3)^2} + \frac{\lambda_2}{(\lambda_1 + \lambda_3)^2}\right],$$

$$I_{12}=\frac{1}{\lambda^2}-\frac{1}{(\lambda_1+\lambda_2)^2},\quad I_{13}=\frac{1}{\lambda^2}+\frac{\lambda_2}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)^2}\ \text{ and}$$

$$I_{23}=\frac{1}{\lambda^2}+\frac{\lambda_1}{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)^2}.$$

The selection procedure based on counts is the selection between the two independent components. Hence selection procedure R_1 is not appropriate to use in Block-Basu model. The asymptotic normal distributions of $(\overline{X}_1-\overline{X}_2)$ and $\left(\hat{\lambda}_2-\hat{\lambda}_1\right)$ can be obtained. By central limit theorem

$$Z_2=\sqrt{n}\,[(\overline{X}_1-\overline{X}_2)-\mu_2]/\sigma_2 \text{ and } Z_3=\sqrt{n}\left[(\hat{\lambda}_2-\hat{\lambda}_1)-\mu_3\right]/\sigma_3$$

have AN (0, 1), where

$$\mu_2=\frac{\lambda\,(\lambda_2-\lambda_1)}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda 3)},\quad \mu_3=(\lambda_2-\lambda_1),$$

$$\sigma_2^2=\frac{(\lambda_1+\lambda_2)^2\left[(\lambda_1+\lambda_3)^2+(\lambda_2+\lambda 3)^2\right]-[\lambda_1\,(\lambda_2+\lambda_3)-\lambda_2\,(\lambda_1+\lambda_3)]^2}{[(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda 3)]^2}$$

and $\sigma_3^2=I^{11}+I^{22}-2\,I^{12}$,

where $I^{ij};\, i,j=1,2,3$ are $(i,j)^{th}$ elements of the inverse of the Fisher information matrix, $I^{-1}(\lambda_1,\lambda_2,\lambda_3)$ is obtained for different values of $(\lambda_1, \lambda_2, \lambda_3)$ and ARE of the selection procedures in BVE of Block-Basu model is given below.

Table (4.1): ARE for different values of $(\lambda_1, \lambda_2, \lambda_3)$ as given in Hanagal (1997).

λ_1	λ_2	λ_3	ARE(R_3,R_2)
0.1	0.16	0.02	0.9840
0.11	0.15	0.02	0.9948
0.12	0.14	0.02	1.0023
0.13	0.14	0.02	1.0041
0.1	0.16	0.03	0.9775
0.11	0.15	0.03	0.9932
0.12	0.14	0.03	1.0060
0.13	0.14	0.03	1.0090
0.1	0.16	0.04	0.9665
0.11	0.15	0.04	0.9920
0.12	0.14	0.04	1.0107
0.13	0.14	0.04	1.0150

Conclusion: It is observed from above table (4.1) that the selection procedure R_3 based on MLE's performs better than the selection procedure R_2 based on sample means when λ_1 closed to λ_2 otherwise the selection procedure R_2 performs better. We also observe that the selection procedures R_2 and R_3 are equally good.

Scope for future study: As a future research work, we propose to compare various selection criteria as given in section 4.2 for various bivariate lifetime distributions by using probability of correct selection. Simulation study will be conducted to compare various procedures.

Appendix:

'C', Program for lower tolerance limits.

```
#include<time.h>
#include<conio.h>
#include<stdlib.h>
#include<math.h>
#include<stdio.h>
void main()
{
FILE *fp;
int i,j,n=100,k,l;
float [100],thet=25.0,theta[5000],u,temp,sum,tavg,L,q=0.05,thetahat;
fp=fopen("rmm1.xls","w");
randomize();
clrscr();
for (l=1;l<=1000;l++)
        {
 for(k=1;k<=n;k++)
        {
         u =(float) random(RAND_MAX)/RAND_MAX;
         t[k]=-thet*log(1-pow(u,1.0/5.0));
printf("\n t[k]=%f",t[k]);
          }
for (i=1; i<=n-1; i++)
```

```
{for(j=i+1; j<=n;j++)
    {  if (t[i]>t[j])
        {   temp=t[i];
            t[i]=t[j];
            t[j]=temp;
    }}      }
sum=0.0;
fprintf(fp,"\n %f",theta[1]);
            t[0]=0.0;
for(k=1; k<=n; k++)
    {
            sum=sum+(n-k+1)*(t[k]-t[k-1])/n;
            fprintf(fp,"\n %f",theta[k]);
    }
            tavg=tavg+sum;
printf("\n simulation No.=%d",l);
        L=L+((-2*n*log(1-q))/(156.4321472))*sum;
        fclose(fp);
            }
printf("\n thetahat=%f",tavg/1000);
            L=L/1000;
            printf("\n L=%f",L);
getch();
    }.
```