CHAPTER-II

FIXED-WIDTH CONFIDENCE INTERVALS :

PURELY SEQUENTIAL PROCEDURES

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2.1: Introduction: -

This chapter deals with general methods of constructing the fixed-width confidence intervals, which are purely sequential in nature. The problem of construction of fixed-width confidence interval for unknown mean of the population with unknown finite variance has been considered by Chow and Robbins(1969). We study this procedure in section (2.2). Also the asymptotic properties of this procedure are established.

Sometimes even though there is no nuisance parameter, the fixed sample size procedure dose not work. For example, the construction of fixed-width confidence interval for variance of the normal population with zero mean. In all such cases a stopping rule can be adopted which will provide a bounded length confidence interval with prescribed coverage probability. Khan(1967) gives the general method of constructing the fixed-width confidence interval of prescribed coverage probability for an unknown parameter of a distribution involving

,possibly, some unkown nuisance parameter. He assumed that distribution involved will be assumed to be known except its parameters. We study this procedure in section (2.3).The asymptotic properties of the procedure are studied in section(2.3.3). The procedure is illustrated with normal distribution and exponential distribution.

In section(2.4), we extend the method due to Khan(1969) to construct a fixed-width confidence interval for $g(\theta)$, a continuous differentiable function of θ. The method is illustrated by constructing a confidence interval for reliability function $R(t)$, with exponential life time distribution. For the same problem, simulation results are reported in the last section (2.5).

2.2: Fixed-width confidence interval for unknown mean of the population:

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. observations from some population with p.d.f. $f(x;\mu,\sigma^2)$. We have to construct a confidence interval of prescribed width 2d (d>0) and prescribed coverage probability $(1-\alpha)$, $(0 < \alpha < 1)$, for the unknown mean μ of the population, when σ^2 the population variance is unknown.

Case 1: Let the variance of the population σ^2 ~~is~~ be known, then the confidence interval for μ with width 2d and the coverage probability $(1-\alpha)$ is constructed as follow.

For any $n \geq 1$, define

$$\bar{X}_n = n^{-1} \sum_{i=1}^{n} X_i$$

and propose the interval

$$I_n = \left\{ \bar{X}_n - d \, , \, \bar{X}_n + d \right\}.$$

Let constant 'a' be such that,

$$(2\pi)^{-1/2} \int_{-a}^{a} \exp(-u^2/2) \, du = (1-\alpha).$$

That is, $$P\left[\, |N(0,1)| \leq a \right] = (1-\alpha).$$

Then,

$$P\left[I_n \ni \mu \right] = P\left[\bar{X}_n - d \leq \mu \leq \bar{X}_n + d \right]$$

$$= P\left[\, | \bar{X}_n - \mu | \leq d \right]$$

$$= P\left[n^{1/2} \, | \bar{X}_n - \mu \, | / \sigma \leq (n^{1/2} d)/\sigma \right]$$

A sample size n is given by ,

$$n = \text{Smallest integer} \geq (a^2 \sigma^2)/\, d^2. \qquad \ldots(2.2.1)$$

From (2.2.1) we have,

$$\lim_{d \to 0} \frac{d^2 n}{a^2 \sigma^2} = 1.$$

Hence, it follows from the central limit theorem that,

$$\lim_{d \to 0} P\left[I_n \ni \mu \right] = (1-\alpha).$$

Case II: Now in this case we assume that σ^2 is not known. It can be easily seen from case(I) above, no fixed sample size procedure exists which gives the desired coverage probability. In this case we define, an estimator of σ^2 to be,

$$V_n = n^{-1}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 + n^{-1} \qquad \ldots(2.2.2)$$

Let $a_1, a_2, \ldots,$ be any sequence of positive constants such that,

$$\lim_{n \to \infty} a_n = a$$

and define,

$$N = \text{smallest } k \geq 1 \text{ such that } V_k \leq d^2k/(a_k^2) \qquad \ldots(2.2.3)$$

Theorem (2.2.1): Chow and Robbins(1965).

Under the sole assumption that $0 < \sigma^2 < \infty$,

$$\lim_{d \to 0} \frac{d^2N}{a^2\sigma^2} = 1 \text{ a.s.} \qquad \ldots(2.2.4)$$

$$\lim_{d \to 0} P\left[I_n \ni \mu \right] = (1-\alpha). \quad \textit{(Asymptotic consistency)} \qquad \ldots(2.2.5)$$

$$\lim_{d \to 0} \frac{d^2E(N)}{a^2\sigma^2} = 1 \text{ a.s.} \quad \textit{(Asymptotic efficiency)} \qquad \ldots(2.2.6)$$

Remark 2.2.1: In case of distribution function of X_i is continuous, definition of V_n in (2.2.2) can be replaced by,

$$V_n = n^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2. \qquad \ldots(2.2.7)$$

Before proving the theorem (2.2.1) , we prove the following lemmas , on which the proof of the theorem is based.

Lemma(2.2.1): Let y_n $(n = 1,2,\ldots)$ be any sequence of the random variables such that $y_n > 0$ a.s. and $\lim_{n\to\infty} y_n = 1$ a.s. Let $f(n)$ be any sequence of constants such that

$$f(n) > 0, \quad \lim_{n\to\infty} f(n) = \infty$$

and

$$\lim_{n\to\infty} \left\{ \frac{f(n)}{f(n-1)} \right\} = ?$$

Incomplete?

and for each $t > 0$, define

$$N = N(t) = \text{Smallest } k \geq 1 \text{ such that } y_k \leq f(k)/t. \qquad \ldots(2.2.8)$$

Thus N is well defined and is a non-decreasing function of t, then

$$\lim_{t\to\infty} N = \infty \text{ a.s.}, \quad \lim_{t\to\infty} E(N) = \infty \qquad \ldots(2.2.9)$$

and

$$\lim_{t\to\infty} \left\{ \frac{f(N)}{t} \right\} = 1 \text{ a.s.} \qquad \ldots(2.2.10)$$

Proof:-

By definition, for each t; t>0

$$N = N(t) = \text{least integer } k\geq 1 \text{ such that } y_k \leq \frac{f(k)}{t}$$

if $t_1<t_2$, then $N(t_1) \leq N(t_2)$ a.s., and $\lim_{t\to\infty} N(t) = \infty$.

Also as $t \to \infty$, $N \to \infty$ a.s., $EN \to \infty$ a.s., therefore $\lim_{t\to\infty} E(N) = \infty$.

To prove (2.2.10), we have,

$$y_N \leq \left\{\frac{f(N)}{t}\right\} = \left\{\frac{f(N)}{f(N-1)}\right\}\left\{\frac{f(N-1)}{f(N)}\right\}$$

$$\leq \left\{\frac{f(N)}{f(N-1)}\right\} y_{N-1}, \text{ Since } y_{N-1} \geq \left\{\frac{f(N-1)}{t}\right\}.$$

Thus, taking limit as $t\to\infty$ we have,

$$\lim_{t\to\infty}\left\{\frac{f(N)}{t}\right\} = 1 \text{ a.s.} \qquad \ldots(2.2.11)$$

Lemma(2.2.2):- If conditions of the lemma (2.2.1) hold and if also $E(\sup_n y_n) < \infty$, then

$$\lim_{t\to\infty} E\left\{\frac{f(N)}{t}\right\} = 1 \text{ a.s.}$$

Proof:- Let $\sup_n y_n = z$ then $E(z) < \infty$.

Choose m such that,

$$\left\{\frac{f(n)}{f(n-1)}\right\} < 2 \qquad ,(n > m).$$

Then for $N > m$ we have,

$$\left\{\frac{f(N)}{t}\right\} = \left\{\frac{f(N)}{f(N-1)}\ \frac{f(N-1)}{t}\right\} < 2y_{N-1} < 2z.$$

On the other hand, if $N < m$,

$$\left\{\frac{f(N)}{t}\right\} = \max_{1 \le n < m} \left\{\frac{f(N)}{t}\right\} \le f(1)+f(2)+....+f(m), \quad t \ge 1.$$

Hence for all $t \ge 1$, we have,

$$\left\{\frac{f(N)}{t}\right\} \le 2z+f(1)+...+f(m).$$

If $E(z) < \infty$

$$\lim_{t\to\infty} \left\{\frac{f(N)}{t}\right\} = E\left[\lim_{t\to\infty} \left\{\frac{f(N)}{t}\right\}\right] = 1.$$

Above result follows by using (2.2.11) and Lebesgue's dominated convergence theorem(1.2.3).

Lemma(2.2.3):- If the conditions of lemma (2.2.1) holds. If $\lim_{n\to\infty} \left\{\frac{f(n)}{n}\right\} = 1$, if for N defined by (2.2.8),

$$\left.\begin{array}{l} E(N) < \infty \quad (\text{all } t > 0) \\ \limsup_{t\to\infty} \left\{E(N\, y_N) \,/\, E(N)\right\} \le 1 \end{array}\right\} \qquad ...(2.2.12)$$

and if there exist a sequence of a constants $g(n)$ such that,

$g(n) > 0$ and $\lim_{n\to\infty} g(n) = 1$, $y_n \ge g(n)y_{n-1}$

then,

$$\lim_{t\to\infty} \frac{E(N)}{t} = 1. \qquad ...(2.2.13)$$

Proof:- for any $\varepsilon > 0$ $(0 < \varepsilon < 1)$, choose m so that,

$$f(n-1) \geq (1-\varepsilon)f(n)$$

$$\geq (1-\varepsilon)n, \qquad \text{for } n \geq m.$$

$$g(n) \geq (1-\varepsilon)$$

and

$$E(Ny_N) \leq (1+\varepsilon)\, E(N), \qquad \text{for } t \geq m.$$

On the set $A = \{ N \geq m\}$ it follows that,

$$\left\{\frac{(1-\varepsilon)^2}{t}\right\} N^2 = (1-\varepsilon)N \left\{\frac{(1-\varepsilon)N}{t}\right\}$$

$$\leq g(N)N \ \frac{f(N-1)}{t} \leq g(N)Ny_{N-t} \leq Ny_N.$$

Hence,

$$\frac{(1-\epsilon)^2}{t}\left[\int_A N\right]^2 \leq \frac{(1-\epsilon)^2}{t}\int_A N^2 \leq \int_A Ny_N \leq E(Ny_N)$$

$$\frac{(1-\epsilon)^2}{t}\int_A N \leq E(Ny_N) \leq \left\{\frac{E(Ny_N)}{\int_A N}\right\}$$

$$\frac{(1-\varepsilon)^2}{t}E(N-m) \leq \left\{\frac{E(Ny_N)}{E(N-m)}\right\} \leq \frac{N\ E(y_N)}{E(N)}$$

from (2.2.9) and (2.2.12), it follows that,

$$(1-\epsilon)^2 \limsup_{t\to\infty} \frac{E(N)}{t} \leq \limsup_{t\to\infty} \frac{E(Ny_N)}{E(N)} \leq 1.$$

This implies,

$$(1-\epsilon)^2 \limsup_{t\to\infty} \frac{E(N)}{t} \le 1. \qquad \ldots(2.2.14)$$

Now let

$$y_n' = \min(1, y_n)$$

then

$$0 \le y_n' \le 1, \; y_n' \le y_n \quad \text{and} \quad \lim_{n\to\infty} y_n' = 1 \text{ a.s.}$$

Define

$$N' = N'(t) = \text{smallest } k \ge 1 \text{ such that } y_k' \le \frac{f(k)}{t}.$$

From lemma (2.2.1),

$$E(\sup_n y_n') \le 1$$

$$1 = \lim_{t\to\infty} \frac{Ef(N)}{t} = \lim_{t\to\infty} \frac{E(N')}{t}$$

but since $y_n' \le y_n$, $N' \le N$.

Hence, $E(N') \le E(N)$.

Thus,

$$\liminf_{t\to\infty} \frac{E(N)}{t} \ge \liminf_{t\to\infty} \frac{E(N')}{t} \qquad \ldots(2.2.15)$$

thus, from (2.2.14) and (2.2.15), we have,

$$\lim_{t\to\infty} \frac{E(N)}{t} = 1.$$

In the following we prove the theorem (2.2.1).

<u>Proof of theorem (2.2.1):</u>

$$\left.\begin{aligned} \text{Let } y_n &= \frac{V_n}{\sigma^2} = \frac{1}{n\sigma^2}\left[\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 + 1\right] \\ f(n) &= \frac{na^2}{a_n^2}, \quad t = \frac{a^2\sigma^2}{d^2} \end{aligned}\right\} \ldots (2.2.16)$$

then (2.2.3) can be written as

$N = N(t)$ = smallest $k \geq 1$ such that $y_k \leq \frac{f(k)}{t}$.

By lemma (2.2.1), we have,

$$1 = \lim_{t\to\infty}\left\{\frac{f(N)}{t}\right\} = \lim_{d\to 0}\left\{\frac{d^2 N}{a^2\sigma^2}\right\} \quad \text{a.s.} \ldots (2.2.17)$$

which is (2.2.4) of the theorem.

Now,

$$P\left[I_n \ni \mu\right] = P\left[\frac{n^{1/2}|X_1+X_2+\ldots+X_n-N\mu|}{\sigma} \leq \frac{N^{1/2}d}{\sigma}\right].$$

From (2.2.17),

$$\frac{dn^{1/2}}{\sigma} \to a \text{ and } \frac{N}{t} \to 1 \text{ in probability.}$$

Hence, from the result of the Anscombe(1952), it follows that as $t \to \infty$,

$$\frac{n^{1/2}|X_1+X_2+\ldots+X_n-N\mu|}{\sigma} \text{ follows } N(0\ ,\ 1).$$

Hence,

$$\lim_{d\to 0} P\left[I_n \ni \mu\right] = (2\pi)^{-1/2}\int_{-a}^{a}\exp(-u^2/2)\,du$$

$$= (1-\alpha).$$

which is (2.2.5) of the theorem.

Now to prove (2.2.6), we have from lemma (2.2.2),

whenever the distribution of X_i is such that,

$$E\left[\sup_n \; n^{-1}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \right] < \infty \qquad \ldots(2.2.18)$$

for then

$$\lim_{t\to\infty} \left\{ \frac{f(N)}{t} \right\} = 1 \text{ a.s.} \qquad \ldots(2.2.19)$$

and from the fact that function f(n) defined by (2.2.16) is n+o(n). It follows from (2.2.18) that,

$$1 = \lim_{t\to\infty} \left\{ \frac{E(N)}{t} \right\} = \lim_{d\to 0} \left\{ \frac{d^2 E\ N}{a^2 \sigma^2} \right\} .$$

For (2.2.18) to hold, it is necessary that fourth moment of X_i to be finite. However using lemma(2.2.3) we will prove that (2.2.6) holds without such restriction.

Fix t> 0, choose m such that $\frac{f(n)}{t} \geq 1$ ($n \geq m$) and choose $\delta > 0$ such that,

$$(n-1)f(n-1) \geq \delta n^2 \; , \; (n\geq 2),$$

and define,for any $r\geq m$,

$$M = \min(N \; , \; r).$$

By Wald's theorem for cumulative sums, we have,

$$E\left[\sum_{i=1}^{M} (X_i - \mu)^2 \right] = EM\ E\ (X_i - \mu)^2$$

$$= EM\ \sigma^2.$$

Hence, by (2.2.16),

$$E(My_M) = 1/\sigma^2\ E\left[\sum_{i=1}^{M}(X_i - \bar{X}_M)^2 + 1\right] \qquad \ldots(2.2.20)$$

$$\leq EM + (1/\sigma^2).$$

Put $g(n) = (n-1)/n$, $(n\geq 2)$, then

$$y_n \geq \frac{1}{n\sigma^2}\sum_{i=1}^{n-1}(X_i - \bar{X}_{n-1})^2 + \frac{1}{n\sigma^2}$$

$$= \left[\frac{n-1}{n}\right] y_{n-1}$$

$$= g(n)\ y_{n-1}.$$

Hence,

$$E(My_M) = \int_{[N>r]} ry_r + \int_{[N\leq r]} Ny_N$$

$$\geq \left[\frac{rf(r)}{t}\right] P[N > r] + \int_{[2\leq N\leq r]} Ny_N$$

$$\geq rP[N>r] + \int_{[2\ \leq N\leq r]} \frac{[Ng(N)f(N-1)]}{t}$$

$$\geq NP[N>r] + \frac{\delta}{t}\int_{[2\leq N\leq r]} N^2.$$

Hence by (2.2.20), we have

$$\int_{[N\leq r]} N \geq \frac{\delta}{t}\int_{[2\leq N\leq r]} N^2 - \left[\frac{1}{\sigma^2}\right]$$

$$\leq \frac{\delta}{t}\left[\int_{[2\leq N\leq r]} N \right]^2 - \left[\frac{1}{\sigma^2} \right].$$

Now letting $r\to\infty$, it follows that

$$E(N) = \lim_{r\to\infty} \int_{[N\leq r]} N \leq \infty.$$

which is first part of (2.2.16).

By Wald's theorem

$$E(Ny_N) \leq E(N) + \left[\frac{1}{\sigma^2} \right],$$

so by (2.2.9), we have,

$$\limsup_{t\to\infty} \left[ENy_N \right] \left[E(N)\right]^{-1} \leq 1,$$

which is second part of the (2.2.16). That is all conditions of the lemma(2.2.3) holds and hence,

$$1 = \lim_{t\to\infty} \left\{ \frac{E(N)}{t} \right\} = \lim_{d\to 0} \left\{ \frac{d^2 EN}{a^2\sigma^2} \right\} \text{ a.s.}$$

which is (2.2.6) of the theorem. Hence the proof of the theorem.

Remark(2.2.2): From theorem (2.2.1) it follows that the purpose of the term n^{-1} in (2.2.2) is to ensure that $y_n = V_n / \sigma^2 > 0$ a.s. This fact is used in lemma (2.2.1) to guarantee that $N \to \infty$ as $t \to \infty$ a.s. If distribution function of X_i is continuous the definition (2.2.7) is equally good, the only change being that

$1/\sigma^2$ in the proof of (2.2.6) disappears.

If the problem is to construct a fixed-width confidence interval for the parameter of interest, not necessarily the mean of the population, we give the general method in the following section, which is due to Khan(1969).

2.3. General Method to Construct a Fixed Width Confidence Interval for the parameter θ:

Let $f(x;\theta_1,\theta_2)$ be a p.d.f. of a random variable X with real valued parameters θ_1 and θ_2, where θ_2 is regarded as a nuisance parameter. We have to construct a fixed-width confidence interval for parameter θ_1 of width 2d (d>0) and with prescribed coverage probability $(1-\alpha)$, $(0<\alpha<1)$, when both θ_1 and θ_2 are unknown.

2.3.1. Assumptions, Notations and Preliminaries:

Let $f_\theta(x_1,x_2,\ldots,x_n)$ be the joint probability density/mass function of random variables $X_1,X_2,\ldots,X_n$. and $\theta = (\theta_1,\theta_2)$. We assume that all regularity conditions for maximum likelihood estimation(MLE) of θ_1, θ_2 are satisfied. That is

(1)The parameter space Θ, is non-degenerate open interval in $\mathbb{R}^2$.

(2)For almost all $x_1,x_2,\ldots,x_n$ and all $\theta \in \Theta$,

$$\frac{\partial f_\theta(x_1,x_2,\ldots,x_n)}{\partial\theta_i}, \quad (i = 1,2)$$

exists, the exceptional set being independent of θ.

(3) $$\frac{\partial}{\partial\theta_i}\int_A f_\theta(x_1,x_2,\ldots,x_n)dx = \int_A \frac{\partial}{\partial\theta_i} f_\theta(x_1,x_2,\ldots,x_n)dx.$$

define t_j?

? (4) $$\frac{\partial}{\partial\theta_i}\int_A t_j f_\theta(x_1,x_2,\ldots,x_n)dx = \int_A \frac{\partial}{\partial\theta_i} t_j f_\theta(x_1,x_2,\ldots,x_n)dx.$$

(5) The elements of matrix

$$\Lambda_\theta = \left\{ l_{ij}(\theta) \right\}$$

exists and are such that Λ_θ is positive definite.

where

$$l_{ij} = -E_\theta\left[\frac{\partial^2 \log f_\theta(x_1,x_2,\ldots x_n)}{\partial\theta_i\partial\theta_j}\right], \quad i,j = 1,2,$$

exists and are such that Λ_θ is positive definite.

Let N denotes the stopping rule and n denotes the fixed size random sample. The Fisher's information matrix is

$$I_n = n\left[l_{ij}\right], \quad i,j = 1,2,$$

and we assume that $\left[l_{ij}\right]$ is positive definite. and

$$\left[l_{ij}\right]^{-1} = \left[\Lambda_{ij}\right] = \Lambda$$

$$\text{i.e. } I^{-1}(n) = \Lambda/n.$$

Let $\hat{\theta}_1(n)$ and $\hat{\theta}_2(n)$ be respectively the MLE's of θ_1 and θ_2 based on the random sample of size n. Since we consider the regular estimation case, the $\hat{\theta}_1(n)$ is asymptotically normal with mean θ_1 and variance λ_{11}/n, where $\lambda_{11} = \lambda_{11}(\theta_1,\theta_2)$, since in general l_{ij}'s are functions of θ_1 and θ_2.

Let $$I_n = \left[\hat{\theta}_1(n) - d \; , \; \hat{\theta}_1(n) + d \right]$$

as a confidence interval of width 2d for θ_1,

and define,

$$n = \text{Smallest integer} \geq \frac{a^2\lambda_{11}(\theta_1,\theta_2)}{d^2} = n_0 \quad \text{(say)}. \qquad \ldots(2.3.1)$$

From (2.3.1) it follows that,

$$\lim_{d\to 0} \left[\frac{d^2 n}{a^2\lambda_{11}(\theta_1,\theta_2)} \right] \geq 1.$$

Hence,

$$\lim_{d\to 0} P\left[I_n \ni \theta_1 \right] = \lim_{d\to 0} P\left[\hat{\theta}_1(n) - d \leq \theta_1 \leq \hat{\theta}_1(n) + d \right]$$

$$= \lim_{d\to 0} P\left[| \hat{\theta}_1(n) - \theta_1 | \leq d \right]$$

$$= \lim_{d\to 0} P\left[\frac{n^{1/2} |\hat{\theta}_1(n) - \theta_1|}{\lambda_{11}(\theta_1,\theta_2)^{1/2}} \leq \frac{d\sqrt{n}}{\lambda_{11}(\theta_1,\theta_2)^{1/2}} \right]$$

$$= P\left[|N(0,1)| \le a' \right], \ a' \ge a,$$

where $a' = \dfrac{d\eta\sqrt{n}}{\lambda_{11}(\theta_1,\theta_2)^{1/2}}$.

Hence

$$\lim_{d\to 0} P\left[I_n \ni \theta_1 \right] \ge 1-\alpha.$$

The n_0 is considered as optimum sample size if $\lambda_{11}(\theta_1, \theta_2)$ are known. This is not justified in strict sense but it will serve as a standard for comparing the stopping random variable in a sequential procedure to be adopted. In some cases n_0 might turn out to be optimum if only θ_2 were known and $\lambda_{11}(\theta_1,\theta_2) = \lambda_{11}(\theta_2)$.

2.3.2: <u>Stopping Rule and its Asymptotic Properties:</u>

Suppose that θ_1 and θ_2 are unknown. In this case fixed n, as determined by (2.3.1), will not be available to gaurantee fixed-width 2d and coverage probability 1-α. Hence, analogous to (2.3.1), we adopt the following sequential rule.

Let m be a given fixed positive integer. Define

N: Starting with n$\ge$ m, stop whenever

$$n \ge \inf \left\{ n \ge m : n \ge \frac{a_n^2 \lambda_{11}(\bar{\theta}_1(n),\bar{\theta}_2(n))}{d^2} \right\} \quad \ldots(2.3.2)$$

and consider

$$I_N = \left[\hat{\theta}_1(N) - d \; , \; \hat{\theta}_1(N) + d \right]$$

as a confidence interval for θ_1.

In the following lemma we prove that the above sequential procedure terminates with probability one under the assumption that $\lambda_{11}(\theta_1,\theta_2) < \infty$., That is the sequential procedure is closed.

Lemma:(2.3.1): Under the assumption $\lambda_{11}(\theta_1,\theta_2) < \infty$, the sequential procedure terminates with probability one.

Proof:- Since we consider a regular estimation case, under the regularity assumption, $\lambda_{11}(\hat{\theta}_1(n),\hat{\theta}_2(n))$ converges to $\lambda_{11}(\theta_1,\theta_2)$ in probability. Hence the right hand side of (2.3.2) tends to n_0 with probability one, which imply that,

$$P[N = \infty] = 0, \text{ That is } P[N < \infty] = 1$$

In the following theorem we prove some asymptotic properties of the sequential rule defined in (2.3.2).

Theorem (2.3.1): Under the assumption $E\left[\sup_n \lambda_{11}(\hat{\theta}_1(n),\hat{\theta}_2(n))\right] < \infty$,

$$\lim_{d\to 0} \frac{N}{n_0} = 1 \text{ a.s.} \qquad \ldots(2.3.3)$$

$$\lim_{d\to 0} P\left[I_N \ni \theta_1 \right] = 1-\alpha. \quad \text{(Asymptotic consistency)} \qquad \ldots(2.3.4)$$

$$\lim_{d\to 0} \frac{E(N)}{n_0} = 1 \text{ a.s.} \quad \text{(Asymptotic efficiency)} \qquad \ldots(2.3.5)$$

Proof: To prove (2.3.3), Let $y_n = \left[\lambda_{11}(\hat{\theta}_1(n), \hat{\theta}_2(n))\right] \left[\lambda_{11}\right]^{-1}$,

$f(n) = \dfrac{na^2}{a_n^2}$ and $t = \dfrac{a^2 \lambda_{11}(\theta_1, \theta_2)}{d^2}$ in lemma (2.2.1),so that the conditions of the lemma are satisfied.

Hence

$$\lim_{t\to\infty} \frac{f(N)}{t} = \lim_{d\to 0} \frac{N}{n_0} = 1 \text{ a.s.}$$

To prove (2.3.4), we observe that, $\dfrac{N(t)}{t} \to 1$ a.s. as $t \to \infty$, and

hence $\dfrac{N(t)}{n_t} \to 1$ a.s. as $t \to \infty$,

where $n_t = [t]$ greatest integer $\le t$.

By taking $Y_n = \hat{\theta}_1(n)$, $F(x) = \Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-u^2/2)\, du$, the condition (C-I) of Anscombe is evidently satisfied by $\hat{\theta}_1(n)$, by taking $w_n = \left[\dfrac{\lambda_{11}}{n}\right]^{1/2}$ and $\theta = \theta_1$. $\hat{\theta}_1(n)$ also satisfies the condition (C-II). Hence by the theorem of Anscombe(1.2.1), it follows that,

$$\left[\frac{n_t}{\lambda_{11}(\theta_1, \theta_2)}\right]^{1/2} \left(\hat{\theta}_1(N_t) - \theta_1\right) \to N(0,1), \text{ as } t \to \infty.$$

Also from (2.3.3), we have,

$$d\left(\frac{N}{\lambda_{11}}\right)^{1/2} \to a \text{ (a.s.), as } d \to 0.$$

Therefore,

$$\lim_{d\to 0} P\left[I_N \ni \theta_1\right] = \lim_{t\to 0} \left\{ \left(\frac{N(t)}{\lambda_{11}}\right)^{1/2} \left|\bar{\theta}_1(N(t))-\theta_1\right| \le d\left(\frac{N(t)}{\lambda_{11}}\right)^{1/2} \right\}$$

$$= P\left[\left|N(0,1)\right| \le a\right]$$

$$= 1 - \alpha,$$

and (2.3.5) follows from lemma (2.2.2).

Note that (2.3.3) and (2.3.4) are universely valid and assumption of $E\left[\sup_n \hat{\lambda}_{11}(n)\right] < \infty$ is required only for validity of (2.3.5). However, in some cases it might be possible to prove (2.3.5) without the assumption $E\left[\sup_n \hat{\lambda}_{11}(n)\right] < \infty$ by using lemma (2.2.3).

In the following we apply the above method to normal and exponential model.

Example(2.3.1): Let $X_1, X_2, \ldots X_n$ be a random sample of size n from $N(\mu,\sigma^2)$,i.e. from distribution with p.d.f.

$$f(x;\mu,\sigma^2) = \begin{cases} (2\pi\sigma^2)^{-1/2} \exp\left\{ -\dfrac{1}{2\sigma^2}(x-\mu)^2 \right\}, & -\infty < x < \infty, \\ & -\infty < \mu < \infty,\ \sigma^2 > 0. \\ 0 & ,\text{otherwise}. \end{cases}$$

where both μ and σ^2 are unknown.

(i)Consider the problem of constructing the fixed-width confidence interval for μ of width 2d and confidence coefficient atleast $1-\alpha$.

Take $\theta_1 = \mu$ and $\theta_2 = \sigma^2$. The Fisher's information matrix $\left(l_{ij}\right)$ $i,j = 1,2$, is obtained in the following.

Here,

$$\log f(x;\theta_1,\theta_2) = -1/2\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2.$$

Therefore,

$$\frac{\partial \log f(x;\theta_1,\theta_2)}{\partial\theta_1}$$

$$= \frac{\partial}{\partial\mu}\left[-1/2\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2\right]$$

$$= \left[\frac{1}{\sigma^2}(x-\mu)\right].$$

Hence

$$l_{11} = E\left[\frac{1}{\sigma^2}(x-\mu)\right]^2$$

$$= \frac{1}{\sigma^4}E(x-\mu)^2$$

$$= \frac{1}{\sigma^2}.$$

$$l_{12} = l_{21} = E\left[\frac{\partial^2 \log f(x;\theta_1,\theta_2)}{\partial\theta_1\partial\theta_2}\right].$$

Now

$$\left[\frac{\partial}{\partial\sigma^2}\left(\frac{\partial log}{\partial\mu} f(x;\theta_1,\theta_2)\right)\right]$$

$$= \left[\frac{\partial}{\partial\sigma^2}\left(\frac{1}{\sigma^2}(x-\mu)\right)\right]$$

$$= \left[-\frac{(x-\mu)}{\sigma^3}\right].$$

Hence

$$l_{12} = l_{21} = E\left[-\frac{(x-\mu)}{\sigma^3}\right]$$

$$= 0.$$

$$l_{22} = E\left[\frac{\partial log f(x;\theta_1,\theta_2)}{\partial\theta_2}\right]^2.$$

Now

$$\left[\frac{\partial}{\partial\sigma^2}\left(-1/2 log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2\right)\right]$$

$$= \left[-\frac{1}{2\sigma^2} - \frac{(x-\mu)^2}{2\sigma^4}\right].$$

Hence

$$l_{22} = E\left[-\frac{1}{2\sigma^2} - \frac{(x-\mu)^2}{2\sigma^4}\right]^2$$

$$= \frac{1}{2\sigma^4}, \quad \text{(on simplification).}$$

This is standard M.Sc level material

Hence the Fisher's information matrix is given by

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A

$$\left\{l_{ij}\right\} = \begin{bmatrix} \theta_2^{-1} & 0 \\ 0 & \frac{1}{2}\theta_2^{-2} \end{bmatrix}$$

and $\lambda_{11}(\theta_1,\theta_2) = \theta_2 = \sigma^2$.

Next we find the MLE of θ_1 and θ_2.

The likelihood function is,

$$L(\theta_1,\theta_2|\underline{x}) = \left(2\pi\ \sigma^2\right)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right\}$$

and the loglikelihood function is given by,

$$log L(\theta_1,\theta_2|\underline{x}) = -\ \frac{n}{2}log(2\pi) - \frac{n}{2}log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

Differentiating with respect to θ_1 and θ_2 and equating to zero, we have,

$$\frac{\partial log L(\theta_1,\theta_2)}{\partial\theta_1} = 0,\ \Rightarrow\ \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = 0$$

$$\Rightarrow \hat{\theta}_1(n) = \hat{\mu} = \bar{x}_n = (\sum_{i=1}^{n}x_i)/n.$$

$$\frac{\partial log L(\theta_1,\theta_2)}{\partial\theta_2} = 0,\ \Rightarrow\ -\frac{n}{2\sigma^2} - \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^4} = 0$$

This is also waste of space!

$$\Rightarrow \hat{\theta}_2(n) = \hat{\sigma}^2 = \sum_{i=1}^{n}(x_i - \bar{x}_n)^2/n.$$

Instead of using $\hat{\theta}_2(n)$ we can use $\hat{\theta}_2^*(n) = \frac{n\hat{\theta}_2(n)}{n-1}$ which is unbiased and also consistent estimator of θ_2. Hence we have the following stopping rule

$$N = \inf\left\{ n \geq 2 \; : \; n \geq \frac{a_n^2 s_n^2}{d^2} \right\}.$$

After terminating the sampling we define confidence interval to be $\left[\bar{X}_N - d, \; \bar{X}_N + d \right]$.

(ii)Now consider the problem of constructing the fixed-width confidence interval for σ^2 of width 2d and confidence coefficient atleast $1-\alpha$.

Take $\theta_1 = \sigma^2$ and $\theta_2 = \mu$, then the Fisher's information matrix is given by

$$\left(\ell_{ij} \right) = \begin{bmatrix} \theta_1^{-2}/2 & 0 \\ 0 & \theta_1^{-1} \end{bmatrix}$$

and $\lambda_{11}(\theta_1, \theta_2) = 2\theta_1^{-1} = 2\sigma^4$.

Hence the stopping rule is given by

$$N = \inf\left\{ n \geq 2 \; : \; n \geq \frac{2a_n^2 s_n^2}{d^2} . \right\} .$$

After terminating the sampling the confidence interval is

$$\left[s_N^2 - d, \; s_N^2 + d \right].$$

<u>Example(2.3.2)</u>:Let $X_1, X_2, \ldots X_n$ be a random sample of size n from exponential distribution with parameter θ, That is from the distribution with p.d.f.,

mean $\frac{1}{\theta}$ failure rate θ

$$
f(x;\theta) = \begin{cases} \theta \exp(-\theta x) & , \ x \geq 0, \ 0<\theta<\infty. \\ 0 & , \ \text{otherwise}. \end{cases}
$$

then the Fisher's information is given by

$$
I(\theta) = - E\left[\frac{\partial^2 \log f(x;\theta)}{\partial \theta^2} \right].
$$

$$
\log f(x;\theta) = \log\theta - \theta x.
$$

$$
\frac{\partial}{\partial\theta} \log f(x;\theta) = \theta^{-1} - x .
$$

$$
\frac{\partial^2}{\partial^2\theta} \log f(x;\theta) = - \theta^{-2}.
$$

Hence,

$$
I(\theta) = - E\left[\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2} \right] = \theta^{-2}.
$$

To find the MLE of θ, the likelihood and loglikelihood functions are respectively given by

$$
L(\theta|\underline{x}) = \theta^n \exp(-\theta \sum_{i=1}^{n} x_i),
$$

$$
\log L(\theta|\underline{x}) = n\log\theta - \theta \sum_{i=1}^{n} x_i .
$$

Differentiating with respect to θ and equating to zero, we have

$$
\frac{\partial \log L(\theta|\underline{x})}{\partial\theta} = 0
$$

$$
\Rightarrow n/\theta - \sum_{i=1}^{n} x_i = 0.
$$

$$
\Rightarrow (\hat{\theta})^{-1} = \sum_{i=1}^{n} x_i \ /n = \bar{x}_n .
$$

Hence from (2.3.2)the stopping rule is given by,

$$N = \inf\left\{ n \geq 2 : n \geq \frac{a_n^2}{(\bar{X}_n^2 d^2)} \right\} \quad \text{and} \quad n_0 = \frac{a^2\theta^2}{d^2}.$$

The hypothesis $E\left[\sup_{n\geq 2} \tilde{\lambda}_{t1}(n)\right] < \infty$ is true in example(2.3.1) which follows from following lemma which is proved from the Winner's theorem. However, the hypothesis is not true in example(2.3.2) and hence (2.3.5) cannot be concluded from lemma(2.2.3).

Now first we state the Winner's theorem (without proof).

Theorem(2.3.2): Let { X_n $n \geq 1$ } be a sequence of i.i.d. random variables with $E|X_n|^r < \infty$ or $E|X_n|^r \log^+|X_n| < \infty$, according as $r>1$ or $r = 1$ then,

$$E\left[\sup_{n\geq 1} \frac{1}{n^r} \left|\sum_{i=1}^{n} X_i\right|^r\right] < \infty$$

and conversely.

Lemma(2.3.2): Under the assumption $0<\sigma^2<\infty$,

$$E\left[\sup_{n\geq 1} S_n^q\right] < \infty,$$

where $$S_n^2 = (n-1)^{-1}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2.$$

Proof: For $q = 2$

$$S_n^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$$

$$= \frac{1}{(n-1)} \sum_{i=1}^{n} X_i^2 - \frac{n}{(n-1)} \bar{X}_n^2.$$

Since,

$$S_n^2 \le \frac{1}{(n-1)} \sum_{i=1}^{n} X_i^2 - \frac{n}{(n-1)} \bar{X}_n^2 + \frac{1}{(n-1)} \bar{X}_n^2 + 2\bar{X}_n^2$$

$$\le X_1^2 + \frac{1}{(n-1)} \sum_{i=2}^{n} X_i^2 + \bar{X}_n^2.$$

Therefore,

$$\sup_{n\ge 2} S_n^2 = X_1^2 + \sup_{n\ge 2} \frac{1}{(n-1)} \sum_{i=2}^{n} X_i^2 + \sup_{n\ge 1} \frac{1}{n^2} (\sum_{i=1}^{n} X_i)^2.$$

Hence,

$$E\left(\sup_{n\ge 2} S_n^2\right) < \infty \text{ if } EX^2 \log^+ |X|^2 < \infty \text{ and } EX^2 < \infty,$$

but $EX^2 \log^+ |X|^2 \le EX^2 < \infty$ and EX^2 are true for the normal distribution with finite variance.

Now we assume that $q > 2$ then

$$S_n^q = (S_n^2)^{q/2} \le \left[\frac{1}{(n-1)} \sum_{i=1}^{n} X_i^2 + \bar{X}_n^2 \right]^{q/2}$$

$$\le 2^{q/2} \left[\frac{1}{(n-1)^{q/2}} \left(\sum_{i=1}^{n} X_i^2 \right)^{q/2} + |\bar{X}_n|^q \right]$$

$$\le 2^{q/2} \left[2^{q/2} \left\{ |X_1|^q + \frac{1}{(n-1)^{q/2}} \left(\sum_{i=2}^{n} X_i^2 \right)^{q/2} \right\} + |\bar{X}_n|^q \right].$$

Therefore

$$\sup_{n\geq 2} S_n^q \leq 2^{q/2}\left[2^{q/2}\left\{ |X_1|^q + \frac{1}{(n-1)^{q/2}}\left(\sum_{i=2}^{n} X_i^2\right)^{q/2}\right\} + \sup_{n\geq 1}\frac{1}{n^q}\left|\sum_{i=1}^{n} X_i\right|^q\right].$$

Hence,

$$E\left(\sup_{n\geq 2} S_n^q\right) < \infty, \text{ if } E|X_1|^q < \infty ,$$

which is true in case of normal distribution with finite variance. This completes the proof of the lemma.

Remark(2.3.1): In case of family of distribution involving single parameter the stopping rule N as defined in(2.3.2) takes the form

$$N = \inf\left\{ n \geq m : n \geq \frac{a_n^2}{d^2 I(\hat{\theta}_n)} \right\},$$

where

$$I(\theta) = -E\left[\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2}\right]$$ and $\hat{\theta}(n)$ is MLE of θ. If Fisher information $I(\theta)$ is independent of θ, the sequential procedure is not necessary, since the bounded length confidence interval of prescribed coverage probability can be based on the normal theory with known variance. In general, sequential procedure to construct fixed-width confidence interval of prescribed coverage probability is not required when $\lambda_{11}(\theta_1,\theta_2) = \lambda_{11}(\theta_2)$ and θ_2 is known.

In the following section we extend method due to Khan(1969) to construct a fixed-width confidence interval for $g(\theta)$, where g is a continuous differentiable function of θ.

2.4: General Method to Construct a Fixed - Width Confidence Interval for the parametric function $g(\theta)$:

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables from distribution with p.d.f. $f(x;\theta)$. We have to construct a fixed-width confidence interval for $g(\theta)$.

Let $\phi = g(\theta)$ be a continuous differentiable function of θ, that is, $g'(\theta) \# 0$ and Let $\theta = g^{-1}(\phi) = g(\phi)$ exist then

$$\log f(x;\theta) = \log f(x; g^{-1}(\phi))$$
$$= \log f(x;\phi).$$

Now by chain rule,

$$\frac{\partial \log f}{\partial \theta} = \frac{\partial \log f}{\partial \phi} \; \frac{\partial \phi}{\partial \theta}$$

and as

$\frac{\partial \phi}{\partial \theta} \# 0$, we have,

$$E\left[\frac{\partial \log f}{\partial \theta}\right] = 0, \text{ if and only if, } E\left[\frac{\partial \log f}{\partial \phi}\right] = 0.$$

Further

$$E\left[\frac{\partial \log f}{\partial \theta}\right]^2 = E\left[\frac{\partial \log f}{\partial \theta}\right]^2 \left[\frac{\partial \phi}{\partial \theta}\right]^2.$$

That is,

$$I(\theta) = I(\phi) \left[\frac{\partial\phi}{\partial\theta}\right]^2 .$$

Hence the Fisher's information for $g(\theta)$ is given by

$$I(\phi) = I(\theta)\left[\frac{\partial\phi}{\partial\theta}\right]^{-2} .$$

Now if $\hat{\theta}$ is MLE of θ then $g(\hat{\theta})$ is the M.L.E. of $g(\theta)$. Then from (2.3.2) the modified stopping rule is given by

$$N = \inf \left\{ n \geq m \ , \ n \geq \frac{a_n^2}{d^2 I(g(\hat{\theta}_n))} \right\} \qquad \ldots(2.4.1)$$

and we propose the interval

$$I_N = \left[\hat{g}_N(\theta)-d \ , \ \hat{g}_N(\theta)+d \right], \text{ for } \theta,$$

wher $\hat{g}_N(\theta)$ is based on N observations.

We illustrate the method by constructing the fixed-width confidence interval for reliability function in the following example.

<u>Example.(2.4.1)</u>:→Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from exponential distribution with mean θ, that is from distribution with p.d.f.

Why change from example 2.3.2 ?

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} \exp(-x/\theta) & , \ x \geq 0, \ \theta > 0. \\ 0 & , \text{ otherwise.} \end{cases}$$

We have to construct a fixed-width confidence interval for the reliability function R(t), that is for $P_\theta[X > t]$.

The Fisher's information for θ is given by

$$I(\theta) = -E\left[\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2}\right].$$

Now,

$$\log f(x;\theta) = -\log\theta - x/\theta.$$

Hence

$$\frac{\partial \log f(x;\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{x}{\theta^2}.$$

and

$$\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2x}{\theta^3} = -\frac{1}{\theta^2}\left[\frac{2x}{\theta} - 1\right].$$

Hence

$$I(\theta) = -E\left[\frac{\partial^2 \mathit{loff}(x;\theta)}{\partial\theta^2}\right] = \frac{1}{\theta^2}E\left[\frac{2x}{\theta} - 1\right]$$

$$= \frac{1}{\theta^2}.$$

Let $R(t) = g(\theta) = P_\theta[X > t] = \exp(-t/\theta)$,

then Fisher's information $I(g(\theta))$ is given by

$$I(g(\theta)) = I(\phi) = I(\theta)\left[\frac{\partial\phi}{\partial\theta}\right]^{-2},$$

where $\phi = g(\theta)$.

Now

$$\left[\frac{\partial\phi}{\partial\theta}\right] = \exp(-t/\theta)(t/\theta^2).$$

Hence,

$$I(g(\theta)) = I(\phi) = (\theta/t)^2 \exp(2t/\theta).$$

From example (2.3.2), similarly we have $\bar{x}_n$ as the MLE of θ. Since R(t) is continuous function of θ, hence MLE of $g(\theta)$ is given by

$$\hat{g}(\theta) = \exp(-t/\bar{x}_n).$$

Hence from (2.4.1), the stopping rule is given by

$$N = \left\{ n \geq 2 \ , \ n \geq \frac{a_n^2 \ t^2}{\bar{X}_n^2 \ d^2 \exp(2t/\bar{X}_n)} \right\}$$

and we propose the interval

$$I_N = \left(\hat{g}_N(\theta) - d \ , \ \hat{g}_N(\theta) + d \right), \text{ for } \theta.$$

In the following section we report the simulation results for example(4.2.1).

2.5. Simulation Results:

An algorithm used for simulation study is provided in appendix (A-I) as well as the corrosponding BASIC program is provided in the appendix (A-II). The following table gives the simulated results for different values of θ, t and d, for α = .05 (table No. 2.5.1 to 2.5.5) and for α = 0.1 (table Nos. 2.5.6 to 2.5.10). The results are based on 500 simulations. R(t) is an actual value of reliability function.

Table 2.5.1

$\theta = 4$ $t = 3$ $R(t) = 0.4723666$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	10.864	10.5295	0.4240461
0.18	13.414	17.28662	0.4308892
0.16	17.16	26.42243	0.4323234
0.14	22.478	42.87348	0.3481049
0.12	30.916	77.41296	0.4364415
0.10	45.206	135.5994	0.4417724
0.08	69.652	408.8706	0.4336736
0.06	128.82	700.8223	0.4524593

Table 2.5.2

$\theta = 7$ $t = 5$ $R(t) = 0.4895417$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	11.484	4.117737	0.4745196
0.18	14.044	6.662064	0.4785355
0.16	17.836	10.65308	0.4836684
0.14	23.794	11.96753	0.4780310
0.12	32.508	16.17395	0.4833384
0.10	46.582	25.24707	0.4872514
0.08	72.706	61.41553	0.4864213
0.06	130.224	76.02735	0.4884756

Table 2.5.3

$\theta = 15$ $t=13$ $R(t)=0.4203504$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	11.916	4.128937	0.3908082
0.18	14.788	6.851044	0.3951396
0.16	18.888	12.33145	0.3975079
0.14	24.808	17.70709	0.4023741
0.12	34.136	28.41345	0.4107821
0.10	49.89	39.97803	0.4094075
0.08	78.596	71.37305	0.4135697
0.06	140.81	84.98633	0.4179109

Table 2.5.4

θ = 27 t = 23 R(t) = 0.4266242

d	E(N)	Var(N)	E[$\hat{R}$(t)]
0.20	11.822	4.342331	0.4078377
0.18	14.9	5.802017	0.4104059
0.16	19.024	9.367432	0.4137769
0.14	25.138	10.57495	0.4144559
0.12	34.314	22.4115	0.4160253
0.10	49.748	32.94043	0.4185827
0.08	79.018	17.17432	0.4242450
0.06	139.66	192.8125	0.4199375

Table 2.5.5

θ = 35 t = 33 R(t) = 0.3895134

d	E(N)	Var(N)	E[$\hat{R}$(t)]
0.20	11.734	5.599228	0.3545625
0.18	14.538	10.26854	0.3509967
0.16	18.736	15.57831	0.3584123
0.14	25.272	16.69403	0.3753188
0.12	34.246	38.83765	0.3724770
0.10	50.382	39.95966	0.3809527
0.08	79.484	80.70557	0.3842320
0.06	143.384	41.5	0.3873111

Table 2.5.6

θ = 4 t = 3 R(t) = 0.4723666

d	E(N)	Var(N)	E[$\hat{R}$(t)]
0.20	8.202	2.045204	0.4518438
0.18	10.014	3.861801	0.4406913
0.16	12.696	5.755585	0.4537945
0.14	16.992	6.227906	0.4575984
0.12	23.162	10.05975	0.4645199
0.10	33.406	15.78528	0.4690286
0.08	52.7	20.35791	0.4683785
0.06	93.466	39.55274	0.4718127

Table 2.5.7

$\theta = 7$ $t = 5$ $R(t) = 0.4895417$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	8.106	2.362763	0.4684128
0.18	9.99	3.425903	0.4747731
0.16	12.66	5.212418	0.4752982
0.14	16.422	7.919892	0.4784027
0.12	22.803	10.05512	0.4784357
0.10	32.29	22.4419	0.4845285
0.08	51.392	21.31055	0.4867553
0.06	91.318	42.60059	0.4877526

Table 2.5.8

$\theta = 15$ $t=13$ $R(t)=0.4203504$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	8.236	2.27276	0.4003424
0.18	10.38	3.459602	0.3927875
0.16	13.14	5.816391	0.3960365
0.14	17.404	7.224793	0.3898887
0.12	23.952	14.50971	0.4003249
0.10	35.028	12.31921	0.4221320
0.08	54.728	34.28589	0.4153286
0.06	98.134	60.97949	0.4196584

Table 2.5.9

$\theta = 27$ $t = 23$ $R(t) = 0.4266242$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	8.324001	2.31102	0.3895110
0.18	10.466	3.152855	0.4052621
0.16	13.166	5.814438	0.3944038
0.14	13.37	7.31308	0.4076867
0.12	24.11	9.529908	0.4157985
0.10	35.192	9.291016	0.4194036
0.08	54.86	16.20434	0.4251566
0.06	98.028	43.63184	0.4253885

Table 2.5.10

$\theta = 35$ $t = 33$ $R(t) = 0.3895134$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	8.526	2.169327	0.4366958
0.18	10.358	4.057839	0.3599142
0.16	12.994	7.321961	0.3569953
0.14	17.166	12.58242	0.3633585
0.12	23.684	21.17212	0.3610205
0.10	35.138	22.20288	0.3749986
0.08	55.038	56.13697	0.3804416
0.06	99.804	57.59278	0.3869181

In the next chapter we consider two-stage sequential procedures to obtain fixed-width confidence interval.