CHAPTER-III

FIXED-WIDTH CONFIDENCE INTERVALS :
TWO-STAGE SEQUENTIAL PROCEDURES

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FIXED - WIDTH CONFIDENCE INTERVALS TWO-STAGE SEQUENTIAL PROCEDURES

3.1. Introduction:

In chapter II we have given ~~the~~ a purely sequential procedure to construct a fixed-width confidence interval for mean of the population with p.d.f. $f(x;\theta)$ with unknown variance. The purely sequential method is asymptotically consistent but fails to acheive the exact consistency. Stein (1945) proposed a two-stage procedure to construct a fixed-width confidence interval for mean μ of a normal distribution when population variance σ^2 is unknown. Mukhopadhyay(1982) developed a two-stage procedure to construct a fixed-width confidence interval for mean of the population along the lines of Stein's two-stage procedure which acheives the exact consistency even without normality. In his method the assumption of the normality is relaxed and replaced by (i)the independence of estimators of the parameter of interest and the nuisance parameter and (ii)the pivotal nature of the estimators in some sense. In section (3.2) we discuss the same and we study asymptotic properties of the same. The procedure is illustrated for normal distribution, negative exponential distribution and the multivariate normal distribution. Since the method is not asymptotically efficient, we consider a modified version of a Stein two-stage procedure which is asymptotically efficient, in section(3.3). We also give asymptotic properties of the modified method. Some of the properties we discuss with negative exponential distribution. In section (3.4), we review the problem of estimating the parameters of an Inverse Gaussian distribution. In section (3.5). some properties of two-stage procedure to construct a fixed-width confidence interval along the lines of Birnbaum and Healy(1960), are reviewed.

3.2: <u>Stein's Procedure and exact consistency:</u>

Let $X_1, X_2, \ldots$ be i.i.d. random variables with p.d.f. $f(x;\theta,\xi)$, where $(\theta,\xi) \in \mathbb{R} \times \mathbb{R}^+$, where $\mathbb{R}$ and $\mathbb{R}^+$ stands for the entire real line, that is $(-\infty,\infty)$, and positive half of the real line, that is $(0,\infty)$, respectively, while support of X may depends on θ alone. Let $T_n = T_n(X_1, X_2, \ldots, X_n)$ and $U_n = U_n(X_1, X_2, \ldots, X_n)$ be estimators of θ and ξ respectively based on sample $X_1, X_2, \ldots, X_n$ and suppose that T_n and U_n satisfy the following conditions.

C-I.For any fixed n, (possibly ≥ 2), T_n is independent of

$(U_2, U_3, \ldots, U_n)$.

C-II.(i) For some $\beta \geq 0$, for a measurable function $g: \mathbb{R}^+ \to \mathbb{R}^+$, the distribution of $\dfrac{n^{\beta}(T_n - \theta)}{g(\xi)}$ does not depend on n, θ and ξ. Let $F(a) = P_{\theta,\xi}\left[\dfrac{n^{\beta}|T_n - \theta|}{g(\xi)} \leq a\right]$, a>0.

(ii) The distribution of $\dfrac{(T_n - \theta)}{g(U_n)}$ does not depend on θ and ξ and there exists a sequence $\{b_n\}$ such that

$$P_{\theta,\xi}\left[\frac{n^{\beta}|T_n - \theta|}{g(U_n)} \leq b_n\right] = 1-\alpha \text{ (say)}, \alpha\in(0,1).$$

Given two preassigned numbers d (d>0) and α ($0<\alpha<1$), we have to construct a confidence interval I_n for θ having width 2d and confidence coefficient atleast $(1-\alpha)$, that is,

$$P_{\theta,\xi}\left[I_n \ni \theta\right] \geq 1-\alpha.$$

Assume that n observations $(X_1, X_2, \ldots, X_n)$ are recorded. We will propose the interval,

$$I_n = \left[T_n - d, T_n + d\right],$$

for θ.

Lower and upper limits of interval will be modified suitably, if support of a random variable X depends on θ. [see example (3.2.2)].

Now,

$$P_{\theta,\xi}\left[I_{n} \ni \theta\right]=P_{\theta,\xi}\left[T_{n} - d \leq \theta \leq T_{n} + d\right]$$

$$= P_{\theta,\xi}\left[\left|T_{n} - \theta\right| \leq d\right]$$

$$=P_{\theta,\xi}\left[\frac{n^{\beta}\left|T_{n} - \theta\right|}{g(\xi)} \leq \frac{n^{\beta}d}{g(\xi)}\right]$$

$$=F\left(\frac{n^{\beta}d}{g(\xi)}\right).$$

Now,

$$P_{\theta,\xi}\left[I_{n} \ni \theta\right] \geq 1-\alpha,$$

if and only if,

$$\frac{n^{\beta}d}{g(\xi)} \geq a, \qquad \text{using C-I (i).}$$

That is, $$n \geq \left(\frac{ag(\xi)}{d}\right)^{1/\beta} = c, \text{ (say)}.$$

Case 1: If ξ is known, we take a sample of size $[c]+1$ and propose the corresponding interval I_n for θ.

Case 2: Suppose that ξ is unknown. In this case, a two-stage procedure similar to that of Stein(1945) is proposed to construct the fixed-width confidence interval for θ. The procedure is as follows, Stage-I: Start an experiment with sample $X_1, X_2, \ldots, X_m$ of size m, $(m \geq 2)$, from $f(x;\theta,\xi)$. Based on these observations compute U_m and b_m and define a stopping random variable as follow,

$$N = \max\left\{ m\ ,\ \left[\left\{ \frac{b_m g(U_m)}{d} \right\}^{1/\beta} \right] + 1 \right\}. \qquad \ldots(3.2.1)$$

Stage-II: Take an independent sample $X_{m+1}, X_{m+2}, \ldots, X_N$ from $f(x;\theta,\xi)$ and propose the interval

$$I_N = \left[T_N - d\ ,\ T_N + d \right], \qquad \ldots(3.2.2)$$

for θ.

In following theorem, we prove that interval defined by (3.2.2) satisfies the definition of exact consistency(1.2.5).

<u>Theorem (3.2.1)</u>: <u>(Exact Consistency)</u>:

For the rule in (3.2.1), under the the assumptions (C-I) and (C-II), for all $(\theta,\xi) \in \mathbb{R}\times\mathbb{R}^+$,

$$P_{\theta,\xi}\left[I_N \ni \theta \right] \geq (1-\alpha). \qquad \ldots(3.2.3)$$

<u>Proof</u>: First we note the basic inequality, using (3.2.1),

$$\left\{ \frac{b_m g(U_m)}{d} \right\}^{1/\beta} \leq N \leq \left\{ \frac{b_m g(U_m)}{d} \right\}^{1/\beta} + m. \qquad \ldots(3.2.4)$$

Now,

$$P_{\theta,\xi}\left[I_N \ni \theta \right] = P_{\theta,\xi}\left[T_N - d \leq \theta \leq T_N + d \right]$$

$$= P_{\theta,\xi}\left[\,|T_N - \theta| \le d\right]$$

$$= \sum_{n=m}^{\infty} P_{\theta,\xi}\left[\,|T_N - \theta| \le d,\ N=n\right]$$

$$= \sum_{n=m}^{\infty} P_{\theta,\xi}\left[\,|T_N - \theta| \le d \;\middle|\; N=n\right] P_{\theta,\xi}\left[N=n\right]$$

$$= \sum_{n=m}^{\infty} P_{\theta,\xi}\left[\,|T_n - \theta| \le d\right] P_{\theta,\xi}\left[N=n\right]. \qquad \ldots(3.2.5)$$

The last step follows from condition (C-I), since the event (N=n) depends only on U_m for any fixed n.

Thus,

$$P_{\theta,\xi}\left[I_N \ni \theta\right] = \sum_{n=m}^{\infty} P_{\theta,\xi}\left[\frac{n^{\beta}|T_n - \theta|}{g(\xi)} \le \frac{dn^{\beta}}{g(\xi)}\right] P_{\theta,\xi}\left[N = n\right]$$

$$= \sum_{n=m}^{\infty} F\left(\frac{dn^{\beta}}{g(\xi)}\right) P_{\theta,\xi}\left[N = n\right]$$

$$= E_{\theta,\xi}\left\{ F\left(\frac{dN^{\beta}}{g(\xi)}\right)\right\}.$$

Since, from (3.2.4), $dN^{\beta} \ge b_m g(U_m)$, we have,

$$P_{\theta,\xi}\left[I_N \ni \theta\right] \ge E_{\theta,\xi}\left\{ F\left(\frac{b_m g(U_m)}{g(\xi)}\right) \right\}$$

$$= E_{\theta,\xi}\left\{ P_{\theta,\xi}\left(|V| \le \frac{b_m g(U_m)}{g(\xi)} \;\middle|\; u_m\right) \right\},$$

where V is independent of U_m and has same distribution as that of

$$\frac{n^{\beta}(T_n - \theta)}{g(\xi)}.$$

Hence, $$P_{\theta,\xi}\left[\, I_N \ni \theta \,\right] \geq P_{\theta,\xi}\left[\, \frac{|V|\, g(\xi)}{g(U_m)} \leq b_m \right] = 1-\alpha.$$

This completes the proof of the theorem.

In the following, the above two-stage procedure is illustrated with some examples.

<u>Example(3.2.1):</u> Let $X_1, X_2, \ldots$ be i.i.d. random variables having normal distribution with mean μ and variance σ^2.

Now let $\theta = \mu$ and $\xi = \sigma$. We have to construct a fixed-width confidence interval for θ.

Choose $T_m = m^{-1}\sum_{i=1}^{m} X_i = \bar{X}_m$ and $U_m = S_m$,

where,

$$S_m^2 = (m - 1)^{-1}\sum_{i=1}^{m} X_i - \bar{X}_m)^2.$$

In this case, let $\beta = 1/2$ and $g(x) = x\ (>0)$, then distribution of

$$\frac{m^{\beta}(T_m - \theta)}{g(\xi)} = \frac{m^{1/2}(\bar{X}_m - \mu)}{\sigma}$$

is N(0,1) and the distribution of

$$\frac{m^{\beta}(T_m - \theta)}{g(U_m)} = \frac{m^{1/2}(\bar{X}_m - \mu)}{S_m}$$

is Student's t with (m-1) degrees of freedom.

Consider

$$I_N = \left[\bar{X}_N - d \, , \, \bar{X}_N + d\right],$$

then from procedure in (3.2.1), we get Stein's two-stage procedure as a special case. Where the sample size N is given by

$$N = \max\left\{ m \, , \, \left[\left\{\frac{b_m S_m}{d}\right\}^2 \right] + 1 \right\}. \qquad \ldots(3.2.6)$$

Here,

$$T_{m-1} = \frac{m^{1/2}(\bar{X}_m - \mu)}{S_m}$$ follows t-distribution with (m-1) degrees of freedom and b_m is such that

$$P\left[\left|T_{m-1}\right| \le b_m\right] = 1-\alpha,$$

Is not this the well known example due to Stein?

implies $b_m = t_{m-1,\alpha/2}$ $100(\alpha/2)\%$ point of Student's t distribution with (m-1) degrees of freedom.

Example (3.2.2): Let $X_1, X_2, \ldots$ are i.i.d. random variables from distribution with p.d.f.

$$f(x;\mu,\sigma)=\begin{cases} \dfrac{1}{\sigma}\exp\left\{-\left\{\dfrac{x-\mu}{\sigma}\right\}\right\}, & \text{for } x>\mu, \sigma>0. \\ 0 \quad , & \text{otherwise.} \end{cases} \qquad \ldots(3.2.7)$$

In this example support of X depends on parameter μ, $(\mu,\sigma) \in \mathbb{R}\times\mathbb{R}^+$.

Now, let $\theta = \mu$ and $\xi = \sigma$.

Choose,

$$T_m = X_{(1)} = \min(X_1, X_2, \ldots, X_m),$$

and $$U_m = \sigma_m = \frac{1}{(m-1)} \sum_{i=1}^{m} (X_i - X_{(1)}).$$

In this case take $\beta = 1$ and $g(x) = x > 0$, then distribution of $\frac{m^{\beta}(T_m - \theta)}{g(\xi)} = \frac{m(X_{(1)} - \mu)}{\sigma}$ follows Chi-square distribution with 2 degrees of freedom and

$$\frac{m^{\beta}(T_m - \theta)}{g(U_m)} = \frac{m(X_{(1)} - \mu)}{\sigma_m}$$

$$= \frac{m(X_{(1)} - \mu)/\sigma}{(m-1) \sum_{i=1}^{m} (X_i - X_{(1)})/\sigma_m}.$$

Since $(m-1) \sum_{i=1}^{m} (X_i - X_{(1)})/\sigma$ has Chi-square distribution with $2(m-1)$ degrees of freedom and $X_{(1)}$ and $\sum_{i=1}^{m} (X_i - X_{(1)})$ are independent, hence distribution of $\frac{m^{\beta}(T_m - \theta)}{g(U_m)}$ has F distribution with 2 and $2(m-1)$ degrees of freedom. Thus, for this example, the sample size is given by

$$N = \max \left\{ m \, , \, \left[\frac{b_m \sigma_m}{2d} \right] + 1 \right\} . \qquad \ldots(3.2.8)$$

In (3.2.8) m is the starting sample size and b_m is such that

$$P\left\{ \frac{m(X_{(1)} - \mu)}{\sigma_m} \le b_m \right\} = 1-\alpha.$$

That is,
$$P\left\{ F_{2,2(m-1)} \le b_m \right\} = 1-\alpha,$$

implies $b_m = F_{2,2(m-1),\alpha}$ i.e. 100α% point of F distribution with 2 and 2(m−1) degrees of freedom.

and finelly we propose the interval,

$$I_N = \left[X_{N(1)} - 2d \, , \, X_{N(1)} \right],$$

for the θ.

Symmetric Multivariate Normal

Example(3.2.3): Let $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)'$ is n-dimensional (n=1,2...) normal with mean vector $\underset{\sim}{\mu} = (\mu, \ldots, \mu)'_{n\times 1}$ and dispersion matrix $\Sigma = \sigma^2 \rho_{ij}$; $\rho_{ii} = 1$ and $\rho_{ij} = \rho$, $(i \neq j = 1, 2, \ldots, n)$. The p.d.f. of $\underset{\sim}{X}$ is given by known ?

$$f(\underset{\sim}{x}) = (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp\left\{ -\frac{1}{2} (\underset{\sim}{x} - \underset{\sim}{\mu})' \Sigma^{-1} (\underset{\sim}{x} - \underset{\sim}{\mu}) \right\}. \quad \ldots(3.2.9)$$

is ρ unknown ?

Suppose $\nu = (\mu, \sigma, \rho) \in \mathbb{R} \times \mathbb{R}^+ \times (-1, 0)$. We wish to construct a confidence interval for μ.

Condition: $\rho \ge -\frac{1}{n-1}$

Let us consider

$$I_n = \left[\bar{X}_n - d \, , \, \bar{X}_n + d \right],$$

as a confidence interval for μ. Then,

$$P_\nu\left\{ I_n \ni \mu \right\} = P_\nu\left\{ \bar{X}_n - d \le \mu \le \bar{X}_n + d \right\}$$

$$= P_\nu\left\{ |\bar{X}_n - \mu| \le d \right\}$$

$$= P_\nu\left\{ \frac{|\bar{X}_n - \mu|}{\left(\frac{\sigma^2[(1-\rho)+n\rho]}{n}\right)^{1/2}} \le \frac{d}{\left(\frac{\sigma^2[(1-\rho)+n\rho]}{n}\right)^{1/2}} \right\}$$

$$\ge P_\nu\left\{ |N(0,1)| \le \frac{d}{\left(\frac{\sigma^2(1-\rho)}{n}\right)^{1/2}} \right\} \qquad \ldots(3.2.10)$$

$$\ge 1-\alpha,$$

if and only if,

$$n \ge \frac{a^2\sigma^2(1-\rho)}{d^2} = C, \text{ (say)},$$

where a is upper $100(\alpha/2)\%$ point of standard normal distribution.

Note that inequality in (3.2.10) is valid since $-1<\rho<0$. X

$\rho \in (\frac{-1}{m-1}, 1)$ $1-\rho>0$

Choose $\theta = \mu$, $\xi = \sigma(1-\rho)^{1/2}$, $T_m = \bar{X}_m$ and $U_m = S_m$ (for $m \ge 2$),

where $S_m^2 = (m-1)^{-1}\sum_{i=1}^{m}(X_i - \bar{X}_m)^2$.

In this case, take $\beta = 1/2$, $g(x) = x$ and define

$$N = \max\left\{ m \;,\; \left[\frac{b_m^2 S_m^2}{d^2}\right]+1 \right\},$$

where, $m \ge 2$ is the first stage sample size. We propose interval

$$I_N = \left[\bar{X}_N - d \;,\; \bar{X}_N + d \right], \text{ for } \mu.$$

We know that, for distribution in (3.2.9), $\bar{X}_m$ and S_m^2 are independent. The distribution of $\bar{X}_m$ is normal with mean μ and variance $[(m+(m-1)\rho)\sigma^2]/m$ and the distribution of $S_m^2/[(1-\rho)\sigma^2]$ is Chi-square with (m-1) degrees of freedom.

Hence,

$\dfrac{m^{1/2}(\bar{X}_m - \mu)}{(S_m)/(m-1)}$ follows t-distribution with (m-1) degrees of freedom.

Now,

$$P_\nu\left\{ I_N \ni \mu \right\} = P_\nu\left\{ \bar{X}_N - d \le \mu \le \bar{X}_N + d \right\}$$

$$= P_\nu\left\{ | \bar{X}_N - \mu | \le d \right\}$$

$$= \sum_n P_\nu\left\{ | \bar{X}_n - \mu | \le d \;\middle|\; N=n \right\} P_\nu\left\{ N = n \right\}$$

$$= \sum_n P_\nu\left\{ \frac{|\bar{X}_n - \mu|}{\left(\dfrac{\sigma^2[(1-\rho)+n\rho]}{n}\right)^{1/2}} \le \frac{d}{\left(\dfrac{\sigma^2[(1-\rho)+n\rho]}{n}\right)^{1/2}} \;\middle|\; N=n \right\}$$

$$\cdot\ P_\nu\left\{ N = n \right\}$$

$$= \sum_n P_\nu\left\{ |N(0,1)| \le \frac{d}{\left(\dfrac{\sigma^2[(1-\rho)+n\rho]}{n}\right)^{1/2}} \;\middle|\; N=n \right\} P_\nu\left\{ N = n \right\}$$

$$\ge \sum_n P_\nu\left\{ |N(0,1)| \le \frac{d}{\left(\dfrac{\sigma^2(1-\rho)}{n}\right)^{1/2}} \;\middle|\; N=n \right\} P_\nu\left\{ N = n \right\}$$

$$= E_{\nu}\left\{2\Phi\left[\frac{d}{\left(\frac{\sigma^2(1-\rho)}{N}\right)^{1/2}}\right] - 1 \,\middle|\, N = n \right\}.$$

Now,

$$N \geq \frac{b_m^2 S_m^2}{d^2}, \text{ this implies } dN^{1/2} \geq b_m S_m.$$

Hence,

$$P_{\nu}\left\{ I_N \ni \mu \right\} \geq E_{\nu}\left\{2\Phi\left[\frac{b_m S_m}{\left(\sigma^2(1-\rho)\right)^{1/2}}\right] - 1 \,\middle|\, S_m \right\}$$

$$= E_{\nu}\left\{P_{\nu}\left[|N(0,1)| \leq \frac{b_m S_m}{\sqrt{\sigma^2(1-\rho)}}\right]\right\}$$

$$= E_{\nu}\left\{P_{\nu}\left[\frac{|N(0,1)|}{S_m / \sqrt{\sigma^2(1-\rho)}} \leq b_m\right]\right\}$$

$$= 1-\alpha.$$

Since, b_m is $100(\alpha/2)\%$ point of Student's t-distribution with $(m-1)$ degrees of freedom. It implies that

$$P_{\nu}\left\{ I_N \ni \theta \right\} \geq 1-\alpha.$$

In the following we state and prove the asymptotic properties of the two-stage procedure described above.

<u>Theorem(3.2.2)</u>: For the rule in (3.2.1), under the

assumptions (C-I) and (C-II), for all $(\theta,\xi)\in\mathbb{R}\times\mathbb{R}^{+}$, as $d \to 0$.

$$\text{(a)}\; dN^{\beta} \to b_m g(U_m). \qquad \ldots(3.2.10)$$

$$\text{(b)}\; E_{\theta,\xi}\left\{\frac{N}{C}\right\} = \left\{\frac{b_m}{a}\right\} E_{\theta,\xi}\left[\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta}\right], \qquad \ldots(3.2.11)$$

Provided $E_{\theta,\xi}\left\{g(U_m)\right\}^{1/\beta}$ is finite.

$$\text{(c)}\; P_{\nu}\left\{I_N \ni \theta\right\} \to 1-\alpha. \qquad \ldots(3.2.12)$$

<u>Proof</u>: From (3.2.4), we have,

$$\left\{\frac{b_m g(U_m)}{d}\right\}^{1/\beta} \leq N \leq \left\{\frac{b_m g(U_m)}{d}\right\}^{1/\beta} + m.$$

It follows that $\lim_{d\to 0} N$ is finite almost surely(a.s.).

Also, we have,

$$\liminf_{d\to 0} dN^{\beta} \geq b_m g(U_m) \quad \text{a.s.} \qquad \ldots(3.2.13)$$

and

$$b_m g(U_m) \geq \limsup_{d\to 0} (N-m)^{\beta} d$$

$$= \limsup_{d\to 0} dN^{\beta}\,(1-m/N)^{\beta}$$

$$= \limsup_{d\to 0} dN^{\beta} \limsup_{d\to 0} (1-m/N)^{\beta}$$

$$= \limsup_{d\to 0} dN^{\beta}.$$

That is,

$$b_m g(U_m) \geq \limsup_{d \to 0} dN^{\beta}. \qquad \ldots(3.2.14)$$

Combining (3.2.13) and (3.2.14), we conclud that,

$$\lim_{d \to 0} dN^{\beta} = b_m g(U_m) \text{ a.s.} \qquad \ldots(3.2.15)$$

This proves that result (a).

To prove (b), again consider the basic inequality

$$\left\{\frac{b_m g(U_m)}{d}\right\}^{1/\beta} \leq N \leq \left\{\frac{b_m g(U_m)}{d}\right\}^{1/\beta} + m.$$

Dividing by $C = \dfrac{ag(\xi)}{d}$, we get,

$$\left\{\frac{b_m g(U_m)}{ag(\xi)}\right\}^{1/\beta} \leq \frac{N}{C} \leq \left\{\frac{b_m g(U_m)}{ag(\xi)}\right\}^{1/\beta} + \left\{\frac{dm}{ag(\xi)}\right\}^{1/\beta}.$$

By taking expectation, on the both side, we get,

$$E_{\theta,\xi}\left\{\frac{b_m g(U_m)}{ag(\xi)}\right\}^{1/\beta} \leq E_{\theta,\xi}\left\{\frac{N}{C}\right\} \leq E_{\theta,\xi}\left\{\frac{b_m g(U_m)}{ag(\xi)}\right\}^{1/\beta} + E_{\theta,\xi}\left\{\frac{dm}{ag(\xi)}\right\}^{1/\beta}.$$

That is ,

$$\left\{\frac{b_m}{a}\right\}^{1/\beta} E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} \leq E_{\theta,\xi}\left\{\frac{N}{C}\right\} \leq \left\{\frac{b_m}{a}\right\}^{1/\beta} E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} + E_{\theta,\xi}\left\{\frac{dm}{ag(\xi)}\right\}^{1/\beta}.$$

By taking the limit as $d \to 0$, we get,

$$E_{\theta,\xi}\left\{\frac{N}{C}\right\} = \left\{\frac{b_m}{a}\right\}^{1/\beta} E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta}.$$

This proves (b), provided $E_{\theta,\xi}[g(U_m)]^{1/\beta}$ is finite.

Now,

$$P_{\theta,\xi}\left\{ I_N \ni \theta\right\} = E_{\theta,\xi}\left\{ F\left[\frac{b_m g(U_m)}{g(\xi)}\right]\right\},$$

which combining with (3.2.15) and dominated convergence theorem (1.2.3) leads to

$$\lim_{d\to 0} P_{\theta,\xi}\left\{ I_N \ni \theta\right\} = E_{\theta,\xi}\left\{ F\left[\frac{b_m g(U_m)}{g(\xi)}\right]\right\}$$

$$= (1-\alpha), \quad \text{using theorm(3.2.1).}$$

This proves (c) of the theorem and hence the proof of the theorem.

There are many examples, where $E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} = 1$, in this case (3.2.11) becomes,

$$E_{\theta,\xi}\left\{\frac{N}{C}\right\} \to \left\{\frac{b_m}{a}\right\}^{1/\beta}, \text{as } d \to 0. \qquad \ldots(3.2.16)$$

Next we see whether $b_m > a$. In addition to conditions (C-I) and (C-II), if some more extra conditions are satisfied we can conclude that $b_m > a$ in fairly general setup.

3.3: Modified Stein-type two-stage procedure:

3.3.1: Normal Distribution:

Stein's two-stage procedure to construct a fixed-width confidence interval is not asymptotically efficient(Ref. Zacks P.No. 558) In this section first we review the Modified two-stage procedure, in brief, to construct a fixed-width confidence interval for mean of normal distribution, which is asymptotically efficient. Further we state asymptotic properties of this procedure.

In Stein's procedure for $N(\mu,\sigma^2)$ distribution, with μ as parameter of interest, the random sample size N is given by

$$N = \max\left\{ n_0, \left[\frac{a^2_{n_0-1} S^2_{n_0}}{d^2}\right]+1 \right\}. \qquad ...(3.3.1)$$

In this procedure $P\left\{ I_N \ni \mu \right\} \geq 1-\alpha$, but, E(N/C) converges to $(a^2_{n_0-1}) / a^2$ as $d \to 0$, which is bigger than one. This means that Stein's procedure is not asymptotically efficient.

To overcome this difficulty, Chow and Robbins(1965) proposed the rule (2.2.3),

$$N = \inf\left\{n: n \geq 2 \text{ and } n \geq \frac{a^2 S_n^2}{d^2} \right\}. \qquad ...(3.3.2)$$

The procedure according to this rule is asymptotically efficient, consistent, but doesnot satisfy the property of exact consistency. That is $P\left\{ I_N \ni \mu \right\} \geq 1-\alpha$. (Ref. theorem(2.2.1)).

To overcome this drawback, we modify the rule (3.3.1) as follows.

$$N = \inf\left\{n: n \geq n_o \text{ and } n \geq \frac{a^2}{d^2}\,(S_n^2 + n^{-1})\right\}. \quad \ldots(3.3.3)$$

The sample size required by (3.3.2) and (3.3.3), for small d, are absolutely close to each other. In case of (3.3.3), we have the lower bound, $N^2 \geq a^2/d^2$ that is $N \geq a/d$. So rule (3.2.3) giving us "asymptotic efficiency" because we take atleast $[a/d]+1$ samples and in turn sample estimate of variance tends to σ^2 as d $\to 0$. We take $S_n^2 + n^{-1}$ as an estimate of σ^2 even if the distribution is continuous, in particular normal.

Now we define a new two-stage procedure as follows.

Let $n_o = \max\left\{2\ ,\ \left[\frac{a}{d}\right]+1\right\}$, then

$$N = \max\left\{ n_o, \left[\frac{a^2_{n_o-1} S^2_{n_o}}{d^2}\right]+1 \right\}. \quad \ldots(3.3.4)$$

The motivation behind (3.3.4) is very simple. The rule (3.3.3) says that we take atleast $[a/d]+1$ samples, so rule (3.3.4) starts with $[a/d]+1$ samples, if d is small.

Theorem(3.3.1):For fixed σ, $0<\sigma<\infty$, the rule in (3.3.4) satisfies the following properties.

(a) $N/C \to 1$ as $d \to 0$.

(b) $\lim_{d \to 0} E(N/C) = 1.$

(c) $P\left\{I_N \ni \mu\right\} \geq 1-\alpha.$

(d) $\lim_{d \to 0} P\left\{I_N \ni \mu\right\} = 1-\alpha.$

Proof:- First we notice the basic inequality from the rule (3.3.5),

$$\frac{a^2_{n0-1} S^2_{n0}}{d^2} \leq N \leq \frac{a^2_{n0-1} S^2_{n0}}{d^2} + \frac{a}{d} + 4. \quad \ldots(3.3.5)$$

Now $n_0 \to \infty$ as $d \to 0$. Hence $S^2_{n0} \to \sigma^2$ a.s. and $a_{n_0-1} \to a$, as $d \to 0$. Using this fact and (3.3.5), we have,

$$N/c \to 1 \text{ a.s., as } d \to 0,$$

which is part (a) of the theorem.

To prove the part (b), dividing (3.3.5) by c and taking expectation and limit as d→0, we have,

$$\lim_{d \to 0} E(N/C) = 1,$$

which is part (b) of the theorem.

To prove (c), consider

$$P[I_N \ni \mu] = P\left\{|\bar{X}_N-\mu| \leq d\right\}$$

$$= \sum_{n=n0}^{\infty} P\left\{|\bar{X}_N-\mu| \leq d \,\middle|\, N=n\right\} P\left\{N = n\right\}$$

$$= \sum_{n=n0}^{\infty} P\left\{|\bar{X}_n-\mu| \leq d\right\} P\left\{N = n\right\}.$$

Since the event {N=n} depends only on S_n^2 for any fixed n.

Hence,

$$P\left\{ I_N \ni \mu \right\} = \sum_{n=n0}^{\infty} P\left\{ \frac{n^{1/2}|\bar{X}_n-\mu|}{\sigma} \le \frac{n^{1/2}d}{\sigma} \right\} P\left\{N=n\right\}$$

$$= \sum_{n=n0}^{\infty} P\left\{ |N(0,1)| \le \frac{n^{1/2}d}{\sigma} \right\} P\left\{ N=n \right\}$$

$$= 2E\left\{ \Phi\left[\frac{N^{1/2}d}{\sigma}\right]-1 \right\}$$

$$\ge 2E\left\{ \Phi\left[\frac{a_{n0-1} S_{n0}}{\sigma}\right]-1 \right\}, \qquad \text{using (3.3.5)}$$

$$= 1-\alpha.$$

Which is the part (c) of the theorem.

Now part (d) follows since,

$$P\left\{ I_N \ni \mu \right\} = 2E\left\{ \Phi\left[\frac{N^{1/2}d}{\sigma}\right]-1 \right\} \to (1-\alpha), \text{ in view of (a).}$$

Hence the theorem.

<u>Remark(3.3.1)</u>: We can define n_0 as,

$$n_0 = \max\left\{2 \text{ , } \left[\left(\frac{a}{d}\right)^{2/(1+\gamma)}\right] +1 \right\}, \text{ for any } \gamma>0,$$

and propose the rule,

$$N = \max\left\{ n_0, \left[\frac{a^2_{n_0-1} S^2_{n_0}}{d^2}\right]+1 \right\}. \qquad \text{...(3.3.6)}$$

For this rule also the properties (a)-(d) holds.

In the following subsection we consider the modified two-stage procedure for non-normal distributions.

3.3.2: Non-normal Distribution:

In this section, the modified two-stage procedure is considered for non-normal set-up, which has all properties as in theorem (2.2.1), is as follow.

Let $X_1, X_2, \ldots, X_n$ be a random sample from d.f. F with mean μ and variance σ^2.

Let $n_0 = \max\left\{2 \ , \ \left[\frac{a}{d}\right]+1 \right\}$, and define,

$$N = \max\left\{ n_0 , \left[\frac{a\ S^2_{n_0}}{d^2}\right]+1 \right\}, \qquad \ldots(3.3.7)$$

and propose the interval $I_N = \left\{ \bar{X}_N - d \ , \ \bar{X}_N + d \right\}$ for the μ.

__Remark(3.3.2):__ The rule like (3.3.6) can also be proposed even in non-normal case and consequently the upper bound for E(N) can be made sharper.

Now, suppose that we are in the same set up as in section (3.2) and we propose a rule like (3.3.6).

Choose a fixed $\gamma \in \mathbb{R}^+$, and let

$$m = \max\left\{2\ ,\ \left[\left(\frac{a}{d}\right)^{1/\beta(1+\gamma)}\right] + 1\right\},$$

and we start a experiment with m observations. Then we define,

$$N = \max\left\{ m\ ,\ \left[\frac{b_m g(U_m)}{d}\right]^{1/\beta} + 1\right\}, \qquad \ldots(3.3.8)$$

and propose the interval,

$$I_N = [T_N - d\ ,\ T_N + d\ ],\ \text{for}\ \theta.$$

The procedure in (3.3.8) looks exactly like the one given in (3.2.1). The major difference in procedure (3.2.1) and (3.3.8) is that, the starting sample size m in procedure (3.3.8) depends on d, the desired length of confidence interval. Also $m = m(d) \to \infty$ as $d \to 0$.The basic inequality in (3.2.4) is changed to

$$\left\{\frac{b_m g(U_m)}{d}\right\}^{1/\beta} \leq N \leq \left\{\frac{b_m g(U_m)}{d}\right\}^{1/\beta} + \left(\frac{a}{d}\right)^{1/\beta(1+\gamma)} + 4. \quad \ldots(3.3.9)$$

The asymptotic properties of the procedure are given in the following theorem.

<u>Theorem (3.3.2):Asymptotic Properties:</u>

Suppose the function $g:\mathbb{R}^+ \to \mathbb{R}^+$ is continuous, U_n is consistent for ξ, that is,

$$E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} = 1.$$

Then under conditions (C-I) and (C-II)

(a) $$P_{\theta,\xi}\left\{ I_N \ni \theta \right\} \geq 1-\alpha. \qquad \ldots(3.3.10)$$

(b) $$\lim_{d\to o} \left(\frac{N}{C}\right) = 1 \text{ in probability.} \qquad \ldots(3.3.11)$$

(c) $$\lim_{d\to o} E_{\theta,\xi} \left(\frac{N}{C}\right) = 1. \qquad \ldots(3.3.12)$$

(d) $$\lim_{d\to o} P_{\theta,\xi}\left\{ I_N \ni \theta \right\} = 1-\alpha. \qquad \ldots(3.3.13)$$

Proof:

To prove (a), consider,

$$\begin{aligned} P_{\theta,\xi}\left[I_N \ni \theta\right] &= P_{\theta,\xi}\left[T_N - d \leq \theta \leq T_N + d\right] \\ &= P_{\theta,\xi}\left[|T_N - \theta| \leq d\right] \\ &= \sum_{n=m}^{\infty} P_{\theta,\xi}\left[|T_N - \theta| \leq d,\ N=n\right] \\ &= \sum_{n=m}^{\infty} P_{\theta,\xi}\left[|T_N - \theta| \leq d \;\middle|\; N=n\right] P_{\theta,\xi}\left[N=n\right] \\ &= \sum_{n=m}^{\infty} P_{\theta,\xi}\left[|T_n - \theta| \leq d\right] P_{\theta,\xi}\left[N=n\right]. \end{aligned}$$

The last step follows from condition (C-I), since the event (N=n) depends only on U_m for any fixed n.

Thus, from (3.2.4), we can write,

$$P_{\theta,\xi}\left[I_N \ni \theta\right] = \sum_{n=m}^{\infty} P_{\theta,\xi}\left[\frac{n^{\beta}|T_n - \theta|}{g(\xi)} \leq \frac{dn^{\beta}}{g(\xi)}\right] P_{\theta,\xi}\left[N = n\right]$$

$$= \sum_{n=m}^{\infty} F\left(\frac{dn^{\beta}}{g(\xi)}\right) P_{\theta,\xi}\left[N = n\right]$$

$$= E_{\theta,\xi}\left\{ F\left(\frac{dN^{\beta}}{g(\xi)}\right)\right\}$$

$$\geq E_{\theta,\xi}\left\{ F\left(\frac{b_m g(U_m)}{g(\xi)}\right) \right\} \quad \text{, using (3.3.9)}$$

$$= E_{\theta,\xi}\left\{P_{\theta,\xi}\left(|V| \leq \frac{b_m g(U_m)}{g(\xi)} \;\middle|\; u_m\right) \right\},$$

where V is independent of U_m ans has same distribution as of $\frac{n^{\beta}(T_n - \theta)}{g(\xi)}$. Hence,

$$P_{\theta,\xi}\left[I_N \ni \theta \right] \geq P_{\theta,\xi}\left[\frac{|V|g(\xi)}{g(U_m)} \leq b_m\right] = 1-\alpha. \qquad \ldots(3.3.14)$$

This completes the proof of the theorem.

Now consider the inequality (3.3.9)

$$\left\{\frac{b_m g(U_m)}{d}\right\}^{t/\beta} \leq N \leq \left\{\frac{b_m g(U_m)}{d}\right\}^{t/\beta} + \left(\frac{a}{d}\right)^{t/\beta(t+\gamma)} +4 .$$

Dividing by C to this inequality, we have,

$$\left\{\frac{b_m g(U_m)}{ag(\xi)}\right\}^{1/\beta} \le \frac{N}{C} \le \left\{\frac{b_m g(U_m)}{ag(\xi)}\right\}^{1/\beta} + \left(\frac{a}{d}\right)^{1/\beta(1+\gamma)}\left\{\frac{d}{ag(\xi)}\right\}^{1/\beta}$$

$$+4\left\{\frac{d}{ag(\xi)}\right\}^{1/\beta}.$$

That is ,

$$\left\{\frac{b_m}{a}\right\}^{1/\beta}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} \le \left\{\frac{N}{C}\right\} \le \left\{\frac{b_m}{a}\right\}^{1/\beta}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} +$$

$$\left(\frac{a}{d}\right)^{-\gamma/\beta(1+\gamma)}\left\{\frac{1}{g(\xi)}\right\}^{1/\beta} +4\left\{\frac{d}{ag(\xi)}\right\}^{1/\beta}. \qquad \ldots(3.3.15)$$

Taking limit as $d \to 0$, we have,

$$\lim_{d\to 0}\left\{\frac{N}{C}\right\} = 1, \qquad \ldots(3.3.16)$$

which proves (b).

To prove (c) take expectation in (3.3.15), we have,

$$\left\{\frac{b_m}{a}\right\}^{1/\beta} E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} \le E_{\theta,\xi}\left\{\frac{N}{C}\right\} \le \left\{\frac{b_m}{a}\right\}^{1/\beta} E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} +$$

$$\left(\frac{a}{d}\right)^{-\gamma/\beta(1+\gamma)}\left\{\frac{1}{g(\xi)}\right\}^{1/\beta} +4\left\{\frac{d}{ag(\xi)}\right\}^{1/\beta}.$$

But U_n is consistent for ξ, that implies,

$$E_{\theta,\xi}\left\{\frac{g(U_m)}{g(\xi)}\right\}^{1/\beta} = 1.$$

Hence by taking limit as $d \to 0$, we have

$$\lim_{d\to 0} E_{\theta,\xi} \left[\frac{N}{C}\right] = 1, \qquad \ldots(3.3.17)$$

which proves (c).

To prove (d) that is $\lim_{d\to 0} P_{\theta,\xi} \left\{ I_N \ni \theta \right\} = 1-\alpha$, we have,

$$P_{\theta,\xi} \left\{ I_N \ni \theta \right\} = E_{\theta,\xi} \left\{ F\left[\frac{dN^{\beta}}{g(\xi)}\right] \right\} .$$

But $\lim_{d\to 0} dN^{\beta} = b_m g(U_m)$ a.s.,

using this property and dominated convergence theorem(1.2.3),

we have,

$$\lim_{d\to 0} P_{\theta,\xi} \left\{ I_N \ni \theta \right\} = E_{\theta,\xi} \left\{ F\left[\frac{b_m g(U_m)}{g(\xi)}\right] \right\}$$

$$= 1-\alpha. \qquad \ldots(3.3.18)$$

Which proves (d) and hence proof of the theorem.

In the following section, we review the problem of estimating the parameters of an Inverse Gaussian distribution in terms of controlling the risk function corrosponding to a suitable zero-one loss function.

3.4: Estimation of Inverse Gaussian parameters:

A random variable X is said to be distributed as inverse Gaussian (IG) with parameters μ and λ, if its p.d.f. is given by

$$f(x;\mu,\lambda) = \begin{cases} \left(\dfrac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\dfrac{\lambda}{2\mu^2}\dfrac{(x-\mu)^2}{x}\right\} & \text{, if } x>0 \\ 0 & \text{, if } x\le 0. \end{cases}$$

where $0 < \mu, \lambda < \infty$. ...(3.4.1)

Inverse Gaussian distribution is perticularly very useful for many long tailed data sets.

Suppose we have a sequence $X_1, X_2, \ldots$ of i.i.d. random variables with inverse Gaussian distribution with above p.d.f. Suppose that n observations $X_1, X_2, \ldots, X_n$ are recorded, usually we take

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \lambda_n = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\bar{X}_n}\right)$$

as estimators for μ and λ^{-1}. It is known that $(\bar{X}_n, \lambda_n)$ forms a complete sufficient statistic for (μ,λ) (Johnson & Kotz(1970) Page No. 143), In estimating μ by $\bar{X}_n$, suppose the loss function is given by

$$L(\mu, \bar{X}_n) = \begin{cases} 1, & \text{if } \dfrac{|\bar{X}_n - \mu|}{(\bar{X}_n)^{1/2}} > 0. \\ 0, & \text{otherwise.} \end{cases} \quad \ldots(3.4.2)$$

This loss function corrosponds to "proportional closeness" in some way. So far we are discussing fixed-width confidence interval procedures. Thus, introducing the loss function like

(3.4.2) is not proper. A reasonable loss function for fixed-width confidence problem we define following 0-1 loss function

$$L(\theta, T_n) = \begin{cases} 1, & \text{if } |T_n - \theta| \geq d \\ 0, & \text{if } |T_n - \theta| < d \end{cases}$$

for the methods in section (2.2) and (2.3). Thus, in section (2.2) and (2.3), given $d>0$ and $\alpha \in (0,1)$, we achieve the exact result $E_{\theta,\xi} L(\theta, T_n) \leq \alpha$. In the following we consider loss function (3.4.2) only.

Given two preassigned numbers d ($d>0$) and α ($0<\alpha<1$), we wish to achieve,

$$E_{\mu,\lambda}\left\{L(\mu, \bar{X}_n)\right\} \leq \alpha. \qquad \ldots(3.4.3)$$

Now,

$$E_{\mu,\lambda}\left\{L(\mu, \bar{X}_n)\right\} = P_{\mu,\lambda}\left\{ \frac{|X_n - \mu|}{(\bar{X}_n)^{1/2}} > d\mu \right\}$$

$$= P_{\mu,\lambda}\left\{ \frac{(n\lambda)^{1/2} |X_n - \mu|}{\mu \quad (\bar{X}_n)^{1/2}} > d(n\lambda)^{1/2} \right\}$$

$$\leq \alpha,$$

if and only if,

$$d(n\lambda)^{1/2} \geq a, \text{ i.e. } n \geq \frac{a^2}{d^2\lambda},$$

where a is upper $100(\alpha/2)\%$ point of $N(0,1)$ distribution, since, $\dfrac{(n\lambda)^{1/2}|X_n-\mu|}{\mu\quad(\bar{X}_n)^{1/2}}$ is distributed as N(0,1).

Thus, the goal of controlling to loss in (3.4.3) can be achieved by fixed sample size procedure, if λ is known, by taking a sample of size $\left[\dfrac{a^2}{d^2\lambda}\right]+1$.

Now we assume that λ is unknown and in this case we propose a two-stage procedure.

Start an experiment with m ($m\geq2$) observations and define

$$N = \max\left\{ m \ , \left[\frac{b_m^2\lambda_m^2}{d^2}\right] + 1 \right\}, \qquad \ldots(3.4.4)$$

where b_m is the $100(\alpha/2)\%$ point of Student's t-distribution with (m−1) degrees of freedom.

Theorem(3.4.1):

For rule in (3.4.4) and loss function in (3.4.2), for all $\mu,\lambda \in \mathbb{R}^+\times\mathbb{R}^+$,

$$E_{\mu,\lambda}\left\{ L(\mu,\bar{X}_n) \right\} \leq \alpha.$$

Proof: First notice that "N = n" depends only on λ_m, which is independent of $\bar{X}_n$ for any fixed $n\geq2$. The proof of this fact follows from Basu's Theorem (1955), since $\bar{X}_n$ is complete sufficient statistic for μ and λ_m is an ancillary statistic for any fixed λ.

Thus,

$$E_{\mu,\lambda}\left\{1-L(\mu,\bar{X}_n)\right\} = P_{\mu,\lambda}\left\{\frac{|\bar{X}_N-\mu|}{\mu(X_N)^{1/2}} \le d\right\}$$

$$= P_{\mu,\lambda}\left\{\frac{(N\lambda)^{1/2}|\bar{X}_N-\mu|}{\mu\ \ \mu(\bar{X}_N)^{1/2}} \le d(N\lambda)^{1/2}\right\}$$

$$= P_{\mu,\lambda}\left\{|N(0,1)| \le d(N\lambda)^{1/2}\right\}$$

$$= E_{\mu,\lambda}\left\{2\Phi\left[d(N\lambda)^{1/2}\right] - 1\right\}$$

$$\ge E_{\mu,\lambda}\left\{2\Phi\left[b_m(\lambda\lambda_m)^{1/2}\right] - 1\right\}, \quad \text{using (3.4.4)}$$

$$= E_{\mu,\lambda}\left\{P_{\mu,\lambda}\left[|Z| \le b_m(\lambda\lambda_m)^{1/2}\middle|\lambda_m\right]\right\},$$

where distribution of Z is $N(0,1)$ which is independent of λ_m, hence,

$$E_{\mu,\lambda}\left\{1-L(\mu,\bar{X}_n)\right\} \ge P_{\mu,\lambda}\left[|Z|\ (\lambda\lambda_m)^{-1/2} \le b_m\right]$$

$$= 1-\alpha,$$

where $|Z|\ (\lambda\lambda_m)^{-1/2}$ follows Student's t-distribution with $(m-1)$ degrees of freedom, since $(m-1)\lambda\lambda_m$ follows chi-square distribution with $(m-1)$ degrees of freedom, the proof of the theorem follows.

In the following section we review some properties of two-stage procedure to construct a fixed-width confidence intervals along the lines of Birnbaum and Healy(1960).

3.5. Birnbaum - Healy type confidence interval:

Birnbaum and Healy (1960) consider the problem of constructing the fixed-width confidence interval but do not consider the problem of achieving the exact consistency. In this section, we consider the extention of method of Birnbaum and Healy for constructing the fixed-width confidence interval for the parameter of interest.

We consider the same set up as in section (3.2), but without the assumption (C-I), when ξ is unknown, we consider the following two-stage procedure.

Start the experiment with the sample of size m (possibly $m \geq 2$), say, $X_1, X_2, \ldots, X_m$. At the second stage, we take a fresh sample $X_{m+1}, X_{m+2}, \ldots, X_{m+N}$. Let $T_N^* = T_N(X_{m+1}, \ldots, X_{m+N})$ and propose the interval,

$$I_N^* = \left\{ T_N^* - d \, , \, T_N^* + d \right\} , \text{ for } \theta.$$

For each fixed n, the event "N = n" depends only on U_m and T_N^* depends only on $(X_{m+1}, \ldots, X_{m+N})$, hence the event "N = n" and T_N^* are independent. Now by using the same technique as in section

(3.2) we can prove that,

$$P_{\theta,\xi}\left\{ I_N^* \ni \theta \right\} \geq 1-\alpha, \qquad \ldots (3.5.1)$$

and

$$\lim_{d\to 0} P_{\theta,\xi}\left\{ I_N^* \ni \theta \right\} = 1-\alpha. \qquad \ldots (3.5.2)$$

Let $N_{BH} = N + m$ and $N_S = N$, where N is defined as

$$N = \max\left\{ m \,, \left[\frac{b_m g(U_m)}{d} \right] + 1 \right\}.$$

Then the asymptotic relative efficiency (ARE) of the Birnbaum-Healy procedure with respect to Stein's procedure(3.2.1), under the condition (C-I) is given by

$$e_{BH,S} = \lim_{d\to 0} \left\{ \frac{E_{\theta,\xi}(N_S)}{E_{\theta,\xi}(N_{BH})} \right\}$$

$$= 1,$$

provided $E_{\theta,\xi}\left\{ \left[g(U_m)\right]^{1/\beta} \right\}$ is finite.

Since ARE being closed to unity, asymptotically these two procedures are equivalent.

<u>Remark(3.5.1)</u>: The results like (3.5.1) and (3.5.2) holds without condition (C-I). However, if condition (C-I) also holds, Stein's procedure beats Birnbaum-Healy procedure in terms of taking fewer samples. requires less number of observations than those required by B-H procedure.

3.6. Conclusions:

(1)The procedures considered above are not applicable to the models, where the only parameter of the model itself is the parameter of interest.

(2)The procedures are not applicable for the problem when $g(\theta_1)$ is the parameter of interest, where g is a continuous known function.

(3)If any of the assumption in condition C-II donot hold we can not use the procedure.

In the following chapter we discuss the problem of fixed-width confidence interval in non-parametric setup.