CHAPTER-IV

FIXED-WIDTH CONFIDENCE INTERVALS : NON-PARAMETRIC SEQUENTIAL PROCEDURES

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4.1. Introduction: -

In chapter II and III we study the construction of fixed-width confidence intervals in parametric setup in the sense that, the form of the distribution is completely known except its parameters. This chapter is devoted to non-parametric sequential interval estimation procedures. In non-parametric setup the functional form of F, the d.f., is completely unknown and it is assumed only that F belongs to suitable family $\mathcal{F}$ of d.f. In section (4.2) we review, in brief, the general method of constructing the non-parametric fixed-width confidence interval along with the asymptotic properties. We obtain fixed-width confidence interval for reliability function. The corresponding simulation results are reported, when F has an expomential distribution with mean θ in section (4.3). The results obtained are compared with the results of the parametric model in section (2.5).

Tahir(1992) has proposed a method to construct a fixed-width confidence interval for correlation coefficient of bivariate normal distribution. We take a review of the results reported by Tahir(1992) in section (4.4).

4.2 A Non-parametric Method to Construct Fixed-width Confidence Interval:

Let $\left\{ X_i ,\ i \geq 1 \right\}$ be a sequence of i.i.d. random variables with d.f. F defined on $\mathbb{R}_p$, for some $p \geq 1$. In parametric model functional form of F is assumed to be known and unknown algebric constants associated with this form are regarded as parameters. In non-parametric setup, F is of unknown form and it is assumed only that F belongs to suitable family $\mathcal{F}$ of d.f. For example $\mathcal{F}$ may be the class of all continuous F on $\mathbb{R}_p$ or all F is (diagonally) symmetric about origin which is taken as location parameter of F. In general in non-parametric formulation we take the parameter

$\theta = \theta(F)$ = a functional of d.f. F. ...(4.2.1)

Thus our parameter of interest is $\theta(F)$.

The objective here is to locate an interval, say I_n, based on sample observatios $X_1, X_2, \ldots, X_n$ of a sample of size n, such that,

(i) $P_F\left\{ I_n \ni \theta(F) \right\} \rightarrow 1-\alpha$, as $d \rightarrow 0$ ...(4.2.2)

(ii) Width of $I_n \leq 2d,\ d > 0$ ...(4.2.3)

Based on sample $(X_1, X_2, \ldots, X_n)$ of size n, let $T_n = T_n(X_1, X_2, \ldots, X_n)$ be a non-parametric estimator of $\theta(F)$. That is T_n is not based on any specific form of F. If F_n is sample (empirical) d.f. based on $(X_1, X_2, \ldots, X_n)$ then F_n is the natural (non-parametric) estimator of F so that one may choose $T_n = \hat{\theta}(F)$ as a natural estimator of θ. This is usually termed as a Von-Mises functional. There are other estimators, which can be considered, such as U-statistic.

We assume that as $n \to \infty$,

$$n^{1/2}(T_n - \theta(F)) \sim N(0, V^2(F)), \qquad \ldots(4.2.4)$$

where $V(F)$, $0 < V(F) < \infty$, is itself is the functional of F. We also assume that there exists a sequence $\{V_n^*\}$ of the estimators $V^2(F)$. (for example, Jackknifed variance estimator).

Note that, (4.2.4) ensures that for large n,

$$P\left[n^{1/2}|T_n - \theta(F)| < Z_{\alpha/2} V(F) \right] \cong 1-\alpha, \qquad \ldots(4.2.5)$$

where $Z_{\alpha/2}$ is upper $100(\alpha/2)\%$ point of the standard normal distribution, By choosing d $(d>0)$ sufficiently small, in (4.2.3), we may set

$$n^* = \min\left\{ n : Z^2_{\alpha/2} V^2(F) d^{-2} \leq n \right\} \qquad \ldots(4.2.6)$$

and obtained that as $d \to 0$,

$$P_F\left\{T_n^* - d \le \theta(F) \le T_n^* + d\right\} \to (1-\alpha), \quad \ldots(4.2.7)$$

so that both (4.2.2) and (4.2.3) holds for an interval,

$$I_n^* = \left\{T_n^* - d \, , \, T_n^* + d\right\}.$$

But definition of n^* in (4.2.6) reveals that n^* depends on unknown F through the V(F). Hence, n^* cannot satisfy (4.2.7) simultaneously for all F belonging to $\mathcal{F}$, the class of d.f. This motivates to develop a sequential procedure to achieve the goal.

In view of assumed consistency of $\{V_n^*\}$, as an estimator of V(F) and (4.2.6), we may consider the stopping rule,

$$N = \inf\left\{ n \ge n_0 \; : \; nd^2 \ge Z^2_{\alpha/2} V_n^* \right\}, \; d>0, \quad \ldots(4.2.8)$$

and define T_N by T_n^* for $n^* = N$. Define,

$$I_N = \left\{ T_N - d \, , \, T_N + d \right\}, \; d>0. \quad \ldots(4.2.9)$$

Note that for I_N (4.1.3) holds good.

Thus, the basic problem is to show that as $d \to 0$,

$$P_F\left\{ I_N \ni \theta(F)\right\} \to (1-\alpha). \quad \ldots(4.2.10)$$

It may also be shown that, under the suitable regularity conditions,

$$\frac{N}{n^*} \longrightarrow 1, \text{ as } d \longrightarrow 0. \qquad \ldots(4.2.11)$$

For example $\frac{N}{n^*} \longrightarrow 1$, as $d \to 0$ or $E\left(\frac{N}{n^*}\right) \longrightarrow 1$, as $d \longrightarrow 0$.

This shows that, for d sufficiently small, N is closed to optimal N and hence the two procedures share a common efficiency.

Suppose that $\{ V_n^* \}$ satisfies the condition that

$$V_n^* \longrightarrow V^2(F), \text{ as } n \to \infty, \qquad \ldots(4.2.12)$$

and the sequence $\{T_n\}$ satisfies the Anscombe's conditions. That is,

$$\max_{m:|m-n|\le \delta_n} \left\{ n^{1/2} |T_m - T_n| \right\} \xrightarrow{P} 0, \text{ as } \delta_n \longrightarrow 0, n \longrightarrow \infty \qquad \ldots(4.2.13)$$

then asymptotic consistency in (4.2.10) holds in the a.s. mode of convergence. In order to establish,

$$\lim_{d \to 0} \left[E\left(\frac{N}{n^*}\right)\right] = 1, \text{ for all } F \in \mathcal{F}, \qquad \ldots(4.2.14)$$

we need some additional conditions on $[V_n^*]$. These conditions are as follow.

(C-I): Suppose that there exists a sequence $\{Z_i\}$ of the i.i.d. random variables such that,

(i) Z_i's are non negative.

(ii) $E(Z_i)$ exists.

(iii) $V_n^* \le ((n-m)^{-1}\sum_{i \le n} Z_i$ for all $n \ge n_0 > m$.

In this setup, we take $E(Z_i) = V^2(F)$ then (4.2.14) holds.

(C-II): Suppose that V_n^* is expression as a linear combination of reversed (sub)martingales, so that,

$$E\left\{ \sup_{n > n_0} V_n^* \right\} < \infty \text{ , for some } n_0 \ge 2, \qquad \ldots (4.2.15)$$

then condition (4.2.14) holds.

(C-III): Suppose that, for some r>1 (not necessarily an integer),

$$E\left\{ \left[n^{1/2} \left| V_n^* - V^2(F) \right| \right]^{2r} \right\} \le C_r < \infty, \text{ for all } n > n_0, \qquad \ldots (4.2.16)$$

then condition (4.2.14) holds.

It is also possible to replace (4.2.16) by the probability inequality,

$$P\left\{ \left| V_n^* - v^2(F) \right| > \varepsilon \right\} \le C_\varepsilon n^{-r}, \text{ for every } n > n_0,$$

where, r>1, (for all $\varepsilon > 0$, $C_\varepsilon < \infty$).

In any case, condition (C-III) is more restrictive than conditions (C-I) and (C-II) and in majority of cases it may be possible to incorporate (C-I) or (C-II) and to avoid the extra moment condition in (C-III). In (4.2.8) often $Z^2_{\alpha/2}$ is replaced by a sequence $\{a_n^2\}$, where $a_n^2 \to Z_{\alpha/2}$ as $n \to \infty$ and the conclusion above remains the same. Also n may be replaced by monotonic function $\psi(n)$ and parallel results holds.

In following example we illustrate the general method of constructing the non-parametric fixed-width confidence interval by constructing the fixed-width confidence interval for reliability function.

Example(4.2.1):

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables from distribution with d.f. F, which is nonnegative and continuous. For some t>0, define

$$R(t) = P_F\left\{ X > t \right\}.$$

Given two preassigned numbers d (d>0) and α ($\alpha \in (0,1)$), we have to construct a fixed-width confidence interval for R(t).

Define

$$Y_i = \begin{cases} 1, & \text{if } X_i > t \\ 0, & \text{if } X_i \le t \end{cases}$$

then $E(Y_i) = P_F\left\{ X_i > t \right\} = R(t)$, that is, Y_i ia an unbiased estimate of R(t).

Now let

$$U_n = \frac{1}{n} \sum_{i=1}^{n} Y_i \text{ , for all } n \ge 1,$$

then $E(U_n) = R(t)$ and $Var(U_n) = R(t)(1-R(t))/n$. Thus $U_n = \hat{R}(t)$ is an unbiased estimate of R(t).

Let

$$U_{in} = \frac{1}{(n-1)} \sum_{\substack{k=1 \\ k \neq i}}^{n} I\left\{ X_k > t \right\}$$ Jackknife version deleting i'th observation.

and $W_{in} = nU_n - (n-1)U_{in}$, $i = 1,2,\ldots,n$.

Then $W_{11}, W_{22}, \ldots, W_{nn}$ are identically distributed random variables and $\bar{W}_n = n^{-1} \sum_{i=1}^{n} W_{in} = U_n$.

Now, let,

$$S_n^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} \left(W_{in} - \bar{W}_n \right)^2,$$

then S_n^2 converges to $R(t)(1-R(t))$, as $n \to \infty$.

Sequential confidence interval for R(t):

Choose a real number 'a' such that $\Phi(a) = (1-(\alpha/2))$, where Φ is standard normal cumulative distribution function and define for $d>0$, Stopping rule as follow.

$$N = \inf\left\{ n \geq 2 \ , \ \hat{\sigma}_n^2 \leq \frac{nd^2}{a_n^2} \right\},$$

and propose the confidence interval for R(t) of form

$$I_N = \left\{ \hat{R}_N(t) - d \ , \ \hat{R}_N(t) + d \right\}.$$

The sequential procedure, given above, satisfies the following properties.

(1) $N \to \infty$ with probability one as $d \to 0$.

(2) $d^2N \to \sigma^2a^2$ with probability one ad $d \to 0$.

(3) $d^2E(N) \to \sigma^2a^2$ as $d \to 0$.

(4) $P\left\{ I_N \ni R(t) \right\} \to (1-\alpha)$ as $d \to 0$.

In the following section we report simulation results for example(4.2.1), by taking $F(x) = 1 - \exp(-x/\theta)$.

4.3: Simulation Results:

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables from exponential distribution, that is from distribution with d.f.

$$F(x) = 1 - \exp(-x/\theta).$$

By following the sequential procedure reported in example (4.2.1), we have the stopping rule,

$$N = \inf\left\{ n\geq 2 \ , \ \hat{\sigma}_n^2 \leq \frac{nd^2}{a_n^2} \right\},$$

where $\hat{\sigma}_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i-\bar{Y}_n)^2$, Y_i is defined in example (4.2.1).

An algorithm to simulate the results is provided in appendix (A-III) as well as the corresponding BASIC program is provided in appendix (A-IV).

Following table gives the simulation results for different values of θ, t and d, for α = 0.05 (Table Nos. 4.3.1 to 4.3.5) and for α = 0.1 (Table Nos. 4.3.6 to 4.3.10). The results are based on 500 simulations. R(t) is an actual value of reliability function.

Table 4.3.1

θ = 4 t = 3 R(t) = 0.4723666

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	22.436	32.12586	0.4687734
0.18	28.18	43.88355	0.4665056
0.16	35.92	69.49378	0.4671592
0.12	63.382	272.408	0.4607748
0.10	91.494	429.3535	0.4685915
0.08	141.2	1231.172	0.4669953
0.06	248.046	4451.461	0.4656341

Table 4.3.2

θ = 7 t = 5 R(t) =0.4895417

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	22.298	32.83319	0.4835564
0.18	28.342	42.84017	0.4940436
0.16	36.326	56.59571	0.4829034
0.14	46.674	117.0039	0.4798471
0.12	62.404	273.4529	0.4765310
0.10	89.946	562.3033	0.4798537
0.08	138.966	1563.803	0.4712047
0.06	250.282	4115.586	0.4956596

Table 4.3.3

θ = 15 t=13 R(t)=0.4203504

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	21.858	31.80858	0.3930772
0.18	27.476	46.62549	0.4051844
0.16	34.456	82.5719	0.4025202
0.14	45.128	126.1317	0.3992306
0.12	62.006	203.2578	0.4039202
0.10	88.95	476.6001	0.4018636
0.08	135.094	1566.707	0.4044673
0.06	238.428	5156.887	0.3955913

Table 4.3.4

θ = 27 t = 23 R(t) = 0.4266242

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	21.638	37.96695	0.3900407
0.18	27.726	45.4469	0.4054049
0.16	34.558	86.31079	0.3994609
0.14	45.686	109.3953	0.3998339
0.12	60.56	304.7024	0.4119602
0.10	84.072	834.6631	0.4027766
0.08	136.93	1364.676	0.4008311
0.06	243.416	4305.094	0.4078117

Table 4.3.5

θ = 35 t = 33 R(t) = 0.3895134

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	21.130	37.59714	0.3576850
0.18	26.674	51.79975	0.3623140
0.16	33.284	92.63135	0.3689130
0.14	43.090	167.2983	0.3598566
0.12	57.826	339.5879	0.3531236
0.10	82.462	731.0020	0.3513969
0.08	132.14	1442.457	0.3663331
0.06	225.622	6309.379)	0.3577929

Table 4.3.6

$\theta = 4$ $t = 3$ $R(t) = 0.4723666$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	16.084	14.74896	0.4591821
0.18	19.482	21.56165	0.4629694
0.16	24.88	35.00964	0.4542325
0.14	32.144	72.93921	0.4736802
0.12	44.44	114.7103	0.4829786
0.10	64.026	225.0772	0.4684698
0.08	98.436	640.4147	0.4681527
0.06	177.254	1631.379	0.4715123

Table 4.3.7

$\theta = 7$ $t = 5$ $R(t) = 0.4895417$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	16.19	12.80968	0.4904125
0.18	20.254	14.7655	0.5034966
0.16	25.486	26.73779	0.4804335
0.14	32.664	64.59497	0.4980311
0.12	44.346	114.7103	0.4829786
0.10	64.642	197.4383	0.4757276
0.08	99.374	590.583	0.4915036
0.06	177.224	1695.697	0.4455253

Table 4.3.8

$\theta = 15$ $t = 13$ $R(t) = 0.4203504$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	15.484	13.62891	0.4051969
0.18	18.916	26.18494	0.3928824
0.16	24.276	38.00787	0.3938440
0.14	31.824	62.48499	0.3980503
0.12	43.238	120.2734	0.4097313
0.10	61.224	286.6781	0.3924973
0.08	95.944	652.0811	0.4063156
0.06	169.192	2233.838	0.4041339

Table 4.3.9

$\theta = 27$ $t = 23$ $R(t) = 0.4266242$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	15.68	15.2816	0.4008805
0.18	19.578	19.64798	0.4170588
0.16	24.43	37.46509	0.3973231
0.14	31.812	64.56464	0.3985085
0.12	42.602	139.1195	0.4120696
0.10	61.888	273.9033	0.4057366
0.08	97.234	537.9024	0.4068598
0.06	170.8	2086.664	0.4120506

Table 4.3.10

$\theta = 35$ $t = 33$ $R(t) = 0.3895134$

d	E(N)	Var(N)	$E[\hat{R}(t)]$
0.20	15.1	16.926	0.3542059
0.18	18.816	24.61014	0.3654689
0.16	23.274	46.66291	0.3585811
0.14	30.580	66.64362	0.3576914
0.12	41.554	125.5231	0.3535610
0.10	59.434	281.0418	0.3121628
0.08	93.264	646.7201	0.3663290
0.06	158.792	3017.735	0.3555142

Remarks(4.3.1):These results are compared with parametric method reported in section (2.5), we observe the following things.

(a) E(N) in the parametric model is considerably small as compared to the non-parametric model. The betterness in the parametric method is achieved without sacrifying the coverage probability.

(b) The variation in the values of N in the parametric method is less as compared to the variation in N in the non-parametric method.

(c)The values of $E(\hat{R}(t))$ in the parametric method are approaching to the actual values of $R(t)$ as d, the width of confidence interval, decreases whereas the $E(\hat{R}(t))$ are not giving any trend in the non-parametric method.

In the following section we review the results reported by Tahir(1992) to construct a fixed-width confidence interval for correlation coefficient of bivariate normal distribution.

4.4. Fixed-Width Sequential Confidence Interval for Correlation Coefficient of Bivariate Normal Distribution:

4.4.1:Introduction:

Why assume means known?

Let $(X_1,Y_1),(X_2,Y_2)...$ be a sequence of independent pairs of random variables and suppose that for each $i = 1,2,...$ (X_i,Y_i) has a bivariate normal distribution $BN(0,0,\sigma_1^2,\sigma_2^2,\rho)$. Thus the joint p.d.f. $f(x_i,y_i)$ of (X_i,Y_i) is given by

$$f(x_i,y_i) = \left\{2\pi\sigma_1^2\sigma_2^2(1-\rho^2)\right\}^{-1/2} \exp\left\{ - \frac{Q(x_i,\ y_i)}{2} \right\} \qquad ...(4.4.1)$$

$$-\infty<(x,y)<\infty$$

$$\sigma_1,\sigma_2>0,|\rho|<1.$$

where,

$$Q(x_i, y_i) = \frac{1}{(1-\rho)^2} \left\{ \frac{x_i}{\sigma_1^2} - 2\rho \frac{x_i y_i}{\sigma_1 \sigma_2} + \frac{y_i}{\sigma_2^2} \right\},$$

where σ_1^2, σ_2^2 and ρ are unknown constants.

Given two numbers d $(d>0)$, and α $(0<\alpha<\alpha)$, we have to construct a sequential fixed-width confidence interval for ρ.

In the following we introduce an unbiased estimator for ρ and find an estimator of its unknown variance.

Lemma(4.4.1): Let (X_1, Y_1) and $(X_2, Y_2$ be independent random variables and suppose that for $i = 1,2$ (X_i, Y_i) follows $BN(0,0,\sigma_1^2,\sigma_2^2,\rho)$. Then

$$P\left[X_1Y_1+X_2Y_2 > 0 \right] = \frac{1+\rho}{2}.$$

Proof: Let $U_i = \frac{X_i}{\sigma_1}$ and $V_i = \frac{Y_i}{\sigma_2}$ for $i = 1,2$.then for each $i=1,2$, (U_i,V_i) has bivariate normal distribution $BN(0,0,1,1,\rho)$.

Consider,

$$4(U_1V_1 + U_2V_2) = \left[(U_1+V_1)^2+(U_2+V_2)^2\right] + \left[(U_1-V_1)^2+(U_2-V_2)^2\right]$$

$$= A - B \text{ (say)}.$$

Where $\frac{A}{2(1+\rho)}$ and $\frac{B}{2(1-\rho)}$ are independent random variables, each of which follows a Chi-square distribution with 2 degrees of freedom.

From these, it follows that,

$$P\left[X_1Y_1+X_2Y_2 > 0\right] = P\left[4(U_1V_1+U_2V_2 > 0\right]$$

$$= P\left[A - B > 0\right]$$

$$= P\left[\frac{A}{B} > 1\right]$$

$$= P\left[F(2,2) > \frac{1-\rho}{1+\rho}\right]$$

$$= \int_{\frac{1-\rho}{1+\rho}}^{\infty} \frac{1}{(1+x)^2}\, dx.$$

Let $(1+x) = t$, then $dx = dt$, then,

$$P\left[X_1Y_1+X_2Y_2 > 0\right] = \int_{\frac{2}{(1+\rho)}}^{\infty} \frac{1}{t^2}\, dt$$

$$= \frac{(1+\rho)}{2}.$$

Now let

$$U_n = \frac{2}{n(n-1)} \sum_{i<j}^{n} I(X_iY_i + X_jY_j)$$

and

$$\tilde{\rho}_n = 2U_n - 1, \text{ for } n \geq 2,$$

where {.} denotes the indicator of set {.}

$$I(X_1Y_1+X_2Y_2) = \begin{cases} 1, & \text{if } X_1Y_1+X_2Y_2>0 \\ 0, & \text{otherwise.} \end{cases}$$

Then by lemma (4.4.1), we have,

$$E\left[U_n\right] = \frac{(1+\rho)}{2}.$$

Hence $\hat{\rho}_n$ is an unbiased estimator of ρ.

Since U_n is U-statistic we can use the properties of U-statistic to construct an estimator of the unknown variance of $\hat{\rho}_n$.

From the result of Hoeffding (1948), we have,

$$\mathrm{Var}(\hat{\rho}_n) = 4\mathrm{Var}(U_n)$$

$$= \frac{16\xi_1}{n} + O\left(\frac{1}{n^2}\right), \text{ as } n \to \infty, \qquad \ldots(4.4.2)$$

where,

$$\xi_1 = \mathrm{Var}\left[\, h\left(\frac{X_1}{\sigma_1}, \frac{Y_1}{\sigma_2}\right)\right], \qquad \ldots(4.4.3)$$

with,

$$h(x,y) = P\left[\frac{X_2 Y_2}{\sigma_1 \sigma_2} > -\, xy\right].$$

Note that ξ_1 depends only on ρ since $\left(\frac{X_1}{\sigma_1}, \frac{Y_1}{\sigma_2}\right)$ has bivariate normal distribution BN(0,0,1,1,ρ).

From (4.4.1), if the terms of order $\frac{1}{n^2}$ are neglected, the problem of estimating Var ($\hat{\rho}_n$) reduces to that of estimating ξ_1.

So let,

$$U_{in} = \frac{2}{(n-1)(n-2)} \sum_{\substack{k<l \\ k,l \neq i}} I(X_k Y_k + X_l Y_l)$$

and

$$W_{in} = nU_n - (n-2)U_{in} \quad \text{for } i=1,2,\ldots,n.$$

Then $W_{11},\ldots,W_{nn}$ are identically distributed random variables and

$$\bar{W}_n = n^{-1} \sum_{i=1}^{n} W_{in} = 2U_n .$$

Now let,

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (W_{in} - \bar{W}_n)^2 \quad \text{for } n\geq 2,$$

then $\quad S_n^2 \to 4\xi_1 \quad$ as $n \to \infty$ [Sen (1977)].

In the following subsection we propose sequential confidence interval for ρ and describe the sequential procedure.

4.4.2 Sequential confidence interval for ρ

Define stopping time by,

$$N = \inf \left\{ n\geq 2 \; : \; \hat{\sigma}_n^2 \leq \frac{nd^2}{a_n^2} \right\},$$

where $\hat{\sigma}_n^2 = 4S_n^2$. Note that $\hat{\sigma}_n^2$ converges to $16\xi_1$ as $n \to \infty$, and $a_n \to a$ as $n \to \infty$.

Finelly construct the confidence interval for ρ of the form

$$I_d = \left[\hat{\rho}_N - d \; , \; \hat{\rho}_N + d \right], \text{ for } \rho.$$

Theorem (4.4.1): Let $\sigma^2 = 16\xi_1$, where ξ_1 is defined by (4.4.2), then

(1) $N \to \infty$ with probability one as $d \to 0$.

(2) $d^2N \to \sigma^2 a^2$ as $d \to 0$.

(3) $d^2E(N) \to \sigma^2 a^2$ as $d \to 0$.

(4) $P[I_d \ni \rho] \to (1-\alpha)$ as $d \to 0$.

Proof: Results (1),(2) and (3) immediately follows from lemma(2.2.1), by letting $y_n = \sigma^{-2}\hat{\sigma}_n^2$, $f(n) = na^2a_n^{-2}$ and $t = a^2\sigma^2d^{-2}$.
To prove(4), we write,

$$P\left[I_d \ni \rho\right] = P\left[\hat{\rho}_N - d \le \rho \le \hat{\rho}_N + d\right]$$
$$= P\left[|\hat{\rho}_N - \rho| \le d\right]$$
$$= P\left[\frac{N^{1/2}|\hat{\rho}_N - \rho|}{\sigma} \le \frac{N^{1/2}d}{\sigma}\right]. \quad ..(4.4.4)$$

Note that $n^{1/2}(\hat{\rho}_n - \rho)$ converges in distribution to a normal random variable with mean zero and variance σ^2 as $n \to \infty$, since $n^{1/2}\left(U_n - \frac{(1+\rho)}{2}\right)$ is asymptotically normal with mean zero and variance $4\xi_1$ as $n \to \infty$.
We finelly use Anscombe's theorem(1.2.1) to obtain that

$$\frac{N^{1/2}(\hat{\rho}_N - \rho)}{\sigma} \rightarrow N(0,1)$$ in distribution as $d \to 0$, since $d^2 N \to a^2\sigma^2$ as $d \to 0$ and conclud from (4.4.4) that $P\left[I_d \ni \rho\right] \to (1-\alpha)$ as $d \to 0$ by (2) and convergence theorem of Cramer(1.2.2).

In the following subsection we describe sequential procedure for constructing fixed-width confidence interval for ρ.

4.4.3 Sequential Procedure:

The sequential procedure for constructing fixed-width confidence interval for ρ can be described as follow.

First boserve the pair (X_1, Y_1) and (X_2, Y_2), then take a pair of observation, one at a time, at each stage n ($n \geq 3$) of the sampling process. Calculate $\hat{\rho}_n$ and an estimate of $\hat{\sigma}^2_n$ of its unknown variance. Check whether,

$n \geq \frac{\hat{\sigma}^2_n a^2_n}{d^2}$ is satisfied or not. After an inequality is satisfied declared that $\left[\hat{\rho}_n - d , \hat{\rho}_n + d\right]$ as a fixed-width confidence interval for ρ.

Remarks(4.4.1):

(1)The procedure described above is valid only if mean vector | known is a null vector.

(2)The author has not reported any simulation results to get an idea of the average sample observations required to obtain the desired level.

(3) It may be interested to look the problem in the parametric setup, that is using Fisher's Z-statistic and its asymptotic distribution, with this the problem of non-null mean vector can also be solved.