CHAPTER - III

SETTING THE CLOCK BACK TO ZERO PROPERTY OF A CLASS OF DISCRETE DISTRIBUTIONS

1. INTRODUCTION :

In Chapter II we have discussed the 'Setting the Clock Back to Zero' property and its characterizations for univariate case. Some distributions possessing SCBZ property were presented and some applications of SCBZ property were discussed.

In this Chapter we discuss the 'Setting the Clock Back to Zero' property and its characterizations for discrete case. Some reliability situations preserved were SCBZ property is are also discussed.

Nair and Mini (1999) have introduced the 'Setting the Clock Back to Zero' property for discrete case. As in the univariate case, this property is an extension of the lack of memory property for discrete case.

Below we present some well known relationships and definitions for discrete case which are used in the squeal.

Let X be a non-negative discrete random variable defined on the support $I^{+} = \{0,1,2,....\}$ having probability mass function

$f(x,\theta)$. Let the family of survival function be denoted by $\{S(x,\theta);\ x = 0,1,2,\ldots,\theta \in \Theta\}$. The failure rate $r(x,\theta)$ and the mean residual life (m.r.l.) function $m(x,\theta)$ are given by the following identities:

$$r(x,\theta) = \frac{f(x,\theta)}{S(x,\theta)} \qquad x \in I^{+} \tag{3.1}$$

$$m(x,\theta) = E_{\theta}(X - x \mid X > x)$$

$$= \frac{1}{S(x+1,\theta)} \sum_{t=x+1}^{\infty} S(t,\theta) \tag{3.2}$$

Salvia and Bollinger (1982) have shown that $r(x,\theta)$ as well as $m(x,\theta)$ determines the distribution uniquely through the following relationships.

$$S(x,\theta) = \prod_{u=0}^{x-1} [1 - r(u,\theta)] \qquad x \in I^{+} \tag{3.3}$$

and

$$S(x,\theta) = \prod_{u=1}^{x-1} \left[\frac{m(u-1,\theta) - 1}{m(u,\theta)}\right] [1 - f(0)] \qquad x \in I^{+} \tag{3.4}$$

where $f(0)$ is determined such that $\sum_x f(x) = 1$. The functions m and r are related through the relation,

$$\frac{m(x-1,\theta) - 1}{m(x,\theta)} = 1 - r(x,\theta) \qquad x \in I^{+} \tag{3.5}$$

with $m(-1,\theta) = \sum_{t=0}^{\infty} S(t,\theta)$.

In section 2, we present the formal definition of SCBZ property for discrete case and some discrete distributions possessing the SCBZ property. In section 3, we discuss some characterizations of SCBZ property in terms of the failure rate function and mean residual life function. In section 4, we show that the SCBZ property of a random variable X is preserved under some reliability operations and we discuss the Closure of SCBZ property in competing risk set-up.

In the next section we present the formal definition of SCBZ property, for discrete case.

2. THE SETTING THE CLOCK BACK TO ZERO PROPERTY:

2.1 DEFINITION :

A non-negative discrete random variable X defined on I^+ having survival function $\{S(x,\theta): x \in I^+, \theta \in \Theta\}$ is said to have the SCBZ property if the survival function satisfies the condition

$$S(x + t,\theta) = S(t,\theta)\, S(x,\theta^*) \quad \forall\, \theta \in \Theta,\ x,t \in I^+. \qquad (3.6)$$

where $\theta^* = \theta^*(t) \in \Theta$, the parametric space.

The interpretations of (3.6) are similar to those of (2.1) given in section 2.1 of Chapter II for continuous case.

Also, as argued for continuous case, this property can be viewed as an extension of univariate lack of memory property of discrete distributions.

2.2. SOME DISCRETE DISTRIBUTIONS POSSESSING THE SCBZ PROPERTY:

In this section we present some discrete distribution functions possessing SCBZ property.

2.2.1. WARING DISTRIBUTION:

The p.d.f. of Waring distribution is

$$f(x,\theta) = \frac{(b)_x}{(a)_x}\left(1 - \frac{b+x}{a+x}\right), \quad x = 0,1,2,\ldots\ldots, \quad a,b > 0.$$

where $(a)_x = a(a+1)\ldots\ldots.(a+x-1)$; with $\theta = (a,b)$ and $\Theta = \{ (a,b);$ $a,b > 0;\ a > b+1\}$.

The survival function is given by

$$S(x,\theta) = \frac{(b)_x}{(a)_x}, \quad x = 0,1,2,\ldots..$$

Therefore,

$$\frac{S(x+t,\theta)}{S(t,\theta)} = \frac{(b)_{x+t}}{(a)_{x+t}} \quad \frac{(b)_t}{(a)_t}$$

$$= \frac{(b+t)_x}{(a+t)_x}$$

$$= S(x,\theta^*), \quad x,t = 0,1,2,3,\ldots..$$

where $\theta^* = (a+t,b+t)$. Hence Waring distribution satisfies the SCBZ property.

2.2.2. NEGATIVE HYPERGOMETRIC DISTRIBUTION:

The p.d.f. of negative hypergometric distribution is

$$f(x,\theta) = \frac{\binom{k+n-x-1}{n-x-1}}{\binom{k+n}{n}}, \quad x = 0,1,2,3,\ldots..n.$$

where $\theta = (k,n)$ and $\Theta = \{(k,n);\ k,n > 0\}$.

The survival function is given by

$$S(x,\theta) = \frac{\binom{k+n-x}{n-x}}{\binom{k+n}{n}}, \quad x = 0,1,2,3,\ldots..n.$$

Therefore,

$$\frac{S(x+t,\theta)}{S(t,\theta)} = \frac{\binom{k+n-x-t}{n-x-t}}{\binom{k+n}{n}} \; \frac{\binom{k+n}{n}}{\binom{k+n-t}{n-t}}$$

$$= \frac{\binom{k+n-x-t}{n-x-t}}{\binom{k+n-t}{n-t}}$$

$$= S(x,\theta^*)$$

where $\theta^*(t) = (k,n-t),\ \forall\ t = 0,1,2,\ldots.n-1$. Hence negative hypergometric distribution satisfies the SCBZ property.

2.2.3. GEOMETRIC DISTRIBUTION:

The p.d.f. of geometric distribution is

$f(x,\theta) = p\, q^x,\ x = 0,1,2,\ldots..\quad 0 \le q \le 1.$

where $\theta = q$ and $\Theta = \{q;\ 0 \leq q \leq 1\}$.

The survival function is given by

$$S(x,\theta) = q^{x},\quad x = 0,1,2,\ldots\ldots \qquad 0 \leq q \leq 1.$$

Therefore,

$$\frac{S(x+t,\theta)}{S(t,\theta)} = \frac{q^{x+t}}{q^{t}},$$

$$= q^{x}$$

$$= S(x,\theta^{*})$$

where $\theta^{*} = \theta = q$, so that geometric distribution satisfies usual LMP and hence SCBZ property.

In the next section we discuss some characterizations of SCBZ property.

3. CHARACTERIZATIONS OF SCBZ PROPERTY:

The SCBZ property (3.6) can also be characterized in terms of the failure rate function r and the mean residual life function m.

A characterization of SCBZ property in terms of the failure rate function is given in the following Theorem.

Theorem 3.1: A non-negative discrete random variable X has SCBZ property if and only if

$$r(x+t,\theta) = r(x,\theta^{*}) \qquad \forall\ x,t \in I^{+},\ \theta,\ \theta^{*} \in \Theta \tag{3.7}$$

where $r(.,\theta)$ is the failure rate function of X.

Proof: Suppose X has SCBZ property. Putting x=x+1 in (3.6), we get

$$S(x+t+1,\theta) = S(t,\theta)\, S(x+1,\theta^{*}) \tag{3.8}$$

Subtracting (3.8) from (3.6) and dividing by S(x+t) using the relation (3.6), we get

$$\frac{S(x+t,\theta)-S(x+t+1,\theta)}{S(x+t,\theta)} = \frac{S(t,\theta)\left[S(x,\theta^{*})-S(x+1,\theta^{*})\right]}{S(t,\theta)\, S(x,\theta^{*})}$$

$$\Rightarrow \frac{S(x+t,\theta)-S(x+t+1,\theta)}{S(x+t,\theta)} = \frac{\left[S(x,\theta^{*})-S(x+1,\theta^{*})\right]}{S(x,\theta^{*})}$$

$$\Rightarrow \frac{f(x+t,\theta)}{S(x+t)} = \frac{f(x,\theta^{*})}{S(x,\theta^{*})}$$

$$\Rightarrow r(x+t,\theta) = r(x,\theta^{*}),\ \forall\ x,t \in I^{+}.$$

Thus (3.7) holds.

Conversely, assume that (3.7) holds. Then by (3.6), we have

$$\frac{S(x+t,\theta)}{S(t,\theta)} = \frac{\prod_{y=0}^{x+t-1}[1-r(y,\theta)]}{\prod_{y=0}^{t-1}[1-r(y,\theta)]}$$

$$= \prod_{y=t}^{x+t-1}[1-r(y,\theta)]$$

$$= \prod_{y=0}^{x-1}[1-r(y+t,\theta)]$$

$$= \prod_{y=0}^{x-1} [1 - r(y,\theta^*)] \qquad \text{(from (3.7))}$$

$$= S(x,\theta^*)$$

Thus X has SCBZ property. ■

The next Theorem gives a characterization of SCBZ property in terms of the mean residual life function.

Theorem 3.2: A discrete random variable X has SCBZ property with support I^+ if and only if

$$m(x+t,\theta) = m(x,\theta^*) \quad x,t \in I^+ \qquad (3.9)$$

where $m(.,\theta)$ is the mean residual life function.

Proof: Let the random variable X has SCBZ property. Then from (3.6), we have

$$\frac{S(u+t+1,\theta)}{S(x+t+1,\theta)} = \frac{S(u+1,\theta^*)S(t,\theta)}{S(x+1,\theta^*)S(t,\theta)}$$

$$= \frac{S(u+1,\theta^*)}{S(x+1,\theta^*)}$$

$$\Rightarrow \quad \frac{1}{S(x+t+1,\theta)} \sum_{u=x}^{\infty} S(u+t+1,\theta) = \frac{1}{S(x+1,\theta)} \sum_{u=x}^{\infty} S(u+1,\theta^*)$$

$$\Rightarrow \quad m(x+t,\theta) = m(x,\theta^*) \qquad \text{(from (3.2))}$$

Conversely, assume that the relation (3.9) holds. From (3.4), we have

$$\frac{S(x+t,\theta)}{S(t,\theta)} = \frac{\prod_{u=1}^{x+t-1}\left[\frac{m((u-1),\theta)-1)}{m(u,\theta)}\right]}{\prod_{u=1}^{t-1}\left[\frac{m((u-1),\theta)-1)}{m(u,\theta)}\right]}$$

$$= \prod_{u=t}^{x+t-1}\left[\frac{m(u-1,\theta)-1}{m(u,\theta)}\right]$$

$$= \prod_{u=0}^{x-1}\left[\frac{m(u+t-1,\theta)-1}{m(u+t,\theta)}\right]$$

$$= \prod_{u=0}^{x-1}\left[\frac{m(u-1,\theta^{*})-1}{m(u,\theta^{*})}\right]$$

$$= \prod_{u=0}^{x-1}[1-r(u,\theta^{*})]$$

$$= S(x,\theta^{*}),$$

Thus X has SCBZ property.

In the next section we show that the SCBZ property of a random variable X is preserved under some reliability operations.

4.CLOSURE UNDER RELIABILITY OPERATIONS.

4.1. DISTRIBUTION OF PARTIAL SUMS:

Let X be a non-negative integer valued random variable with probability mass function $f(x,\theta)$, survival function $S(x,\theta)$ and finite mean μ. Then a distribution can be derived with probabilities proportional to the complement of the parent cumulative distribution function of X. The distribution is

$$g(x,\theta) = \mu^{-1} \sum_{t=x+1}^{\infty} f(t,\theta)$$

$$= \mu^{-1} S(x+1,\theta) \qquad (3.10)$$

and is called as the distribution based on partial sums and a random variable Y with probability density (3.10) is called the random variable corresponding to the partial sums.

The SCBZ property (3.6) of the parent distribution preserves in the distribution based on the partial sums and vice versa. This is proved in the following Theorem.

Theorem 3.3: The non-negative integer valued random variable X has SCBZ property if and only if Y, the random variable corresponding to the partial sums has SCBZ property.

Proof :IF PART: Let X has SCBZ property. Then we have,

$$G(x+t,\theta) = \sum_{u=x+t}^{\infty} g(u,\theta)$$

$$= \mu^{-1} \sum_{u=x+t}^{\infty} S(u+1,\theta)$$

$$= \mu^{-1} \sum_{u=o}^{\infty} S(x+t+u+1,\theta)$$

$$= \mu^{-1} \sum_{u=o}^{\infty} S(t,\theta)\ S(x+u+1,\theta^{*}) \qquad \text{(from (3.6))}$$

$$= \mu^{-1} S(t,\theta) \sum_{u=0}^{\infty} S(x+u+1,\theta^{*}) \tag{3.11}$$

Also $\mu^{*} S(t,\theta) = S(t,\theta) \sum_{u=1}^{\infty} S(u,\theta^{*})$ (from (3,10) $\mu^{*} = \sum_{u=1}^{\infty} S(u,\theta^{*})$)

$$= S(t,\theta) \sum_{u=0}^{\infty} S(u+1,\theta^{*})$$

$$= \sum_{u=0}^{\infty} S(t,\theta)\, S(u+1,\theta^{*})$$

$$= \sum_{u=0}^{\infty} S(t+u+1,\theta) \qquad \text{by (3.6)}$$

Therefore

$$S(t,\theta) = (\mu^{*})^{-1} \sum_{u=0}^{\infty} S(t+u+1,\theta)$$

Putting the value of $S(t,\theta)$ in equation (3.11), we get

$$G(x+t,\theta) = (\mu^{*})^{-1} \sum_{u=0}^{\infty} S(t+u+1,\theta)\ \mu^{-1} \sum_{u=0}^{\infty} S(x+u+1,\theta^{*})$$

$$= G(t,\theta)\, G(x,\theta^{*}) \tag{3.12}$$

This shows that has SCBZ property.

Conversely, Let us assume that Y has SCBZ property.

SCBZ property implies (3.12). Then by Theorem 3.1, we have

$$r_y(x+t,\theta) = r_y(x,\theta^{*}) \quad \forall \quad x,t \in I^{+} \tag{3.13}$$

Where $r(.,\theta)$ is the failure rate of Y. Gupta (1979) have shown that $r_Y(x,\theta) = \dfrac{1}{m(x,\theta)}$, Therefore equation (3.13) becomes,

$$\frac{1}{m(x+t,\theta)} = \frac{1}{m(x,\theta^*)}$$

$$\Rightarrow \quad m(x+t,\theta) = m(x,\theta^*)$$

Hence by Theorem 3.2, X has SCBZ property.

4.2. Closure Under Competing risk set-up:

In this section we discuss the closure of SCBZ property in competing risk set-up. The result is proved in the following Theorem.

Theorem 3.4: Let $X_1,\ldots,X_n$ denote the lifetimes of an organism under risk 1, risk 2,...,risk n respectively which areassumed to be independently distributed. Let $Y = \min(X_1,\ldots,X_n)$ denote the observed lifetime of the organism. If the survival distributions of $X_1, X_2,\ldots,X_n$ have the SCBZ property, then the survival distribution of Y also has SCBZ property.

Proof: Let survival functions under the n risks be $S_{X_1}(y,\theta_1)$, $S_{X_2}(y,\theta_2),\ldots,S_{X_n}(y,\theta_n)$ respectively. Then the survival function of $Y = \min(X_1,\ldots,X_n)$ is given by

$$S_Y(y,\theta) = P(Y \geq y)$$

$$= P(\min(X_1,\ldots,X_n) \geq y)$$

$$= P(X_1 > y,\ X_2 > y,\ldots,X_n > y)$$

$$= P(X_1 > y)\, P(X_2 > y) \ldots P(X_n > y) \qquad \text{(Since } X_i \text{ are independently distributed)}$$

$$= \prod_{i=1}^{n} P(X_i > y)$$

$$= \prod_{i=1}^{n} S_i(y, \theta_i),$$

Here $\theta = (\theta_1, \ldots \theta_n)$, is the vector parameter for the observed life length Y. Now consider

$$\frac{S_Y(y+s, \theta)}{S_Y(s, \theta)} = \frac{\prod_{i=1}^{n} S_i(y+s, \theta_i)}{\prod_{i=1}^{n} S_i(s, \theta_i)}$$

$$= \prod_{i=1}^{n} \frac{S_i(y+s, \theta_i)}{S_i(s, \theta_i)}$$

$$= \prod_{i=1}^{n} S_i(y, \theta_i^{*}), \qquad \text{(Since } X_i \text{ has SCBZ property)}$$

$$= S_Y(y, \theta^{*}) \qquad y \in I^{+}.$$

where $\theta^{*} = (\theta_1^{*}, \theta_2^{*} \ldots \theta_n^{*})$. Note that $\theta_i^{*} \in \Theta_i$, so that $\underline{\theta}^{*} \in (\Theta_1 \times \Theta_2 \times \ldots \Theta_n)$. Thus the family of life distributions of the observed life length of the organism also possesses the SCBZ property.

—— O X O ——