APPENDIX A

LEMMA A . 1 : A 2^{k-p} design of even resolution $R = 2l$ must contain at least

$$\sum_{i=0}^{l-1} \binom{k}{i} + \binom{k-1}{l-1} \qquad (A.1)$$

runs.

PROOF : In the design matrix, let $X_1^{(1)}, ...X_k^{(1)}$ be the column vectors associated with the k main effects, $X_1^{(2)}, ...X_k^{(2)}$ be the column vectors associated with the $\binom{k}{2}$ two-factor interactions, . . . , let $X_1^{(l-1)}, ...X_k^{(l-1)}$ be the column vectors associated with the $\binom{k}{l-1}$ $(l-1)^{th}$-factor interactions and the matrix X be given by,

$$X = \left[X_1^{(1)}, ...X_k^{(1)}, X_1^{(2)}, ...X_{\binom{k}{2}}^{(2)}, ...X_1^{(l-1)}, ...X_{\binom{k}{l-1}}^{(l-1)}\right]$$

Let $Z_0, Z_1, ...Z_{\binom{k-1}{l-1}}$ represent the column vectors associated with the grand mean and the interactions of the first factor with the $\binom{k-1}{l-1}$ $(l-1)^{th}$ factor interactions which do not include the first factor. Let the matrix Z be given by,

$$Z = \left[Z_0, Z_1, ...Z_{\binom{k-1}{l-1}}\right].$$

The columns of the matrix $\left[X, \quad Z\right]$ are orthogonal to each other and linearly independent.

Suppose the design contains $N \leq \sum_{i=0}^{l-1} \binom{k}{i} + \binom{k-1}{l-1}$ runs. Let $s = N - \sum_{i=1}^{l-1} \binom{k}{i}$. Then select any vectors $W_1, ...W_s$ such that the matrix $\left[X, \quad W\right]$ is of rank N.

$\Rightarrow N \geq \sum_{i=1}^{l-1} \binom{k}{i}$. Since $\begin{bmatrix} X & W \end{bmatrix}$ is of full rank N there exist matrices H_1 and H_2 such that $Z = XH_1 + WH_2$.

The model for underlying design is, $Y = X\beta_1 + W\beta_2 + Z\beta_3 + \epsilon$ where $\underline{Y}$ is the $N \times 1$ vector of responses, $\beta_1 = \left(\sum_{i=0}^{l-1} \binom{k}{i} \times 1\right)$ vector of parameters to be estimated and $\beta_2 = \left(N - \sum_{i=0}^{H} \binom{k}{i}\right) \times 1$, $\beta_3 = \left(\binom{k-l}{l-1} \times 1\right)$ vectors of parameters not to be estimated. Since $E(\hat{\beta}_1) = \beta_1$ and $\hat{\beta}\,1 = JY$ for some matrix J, we have

$EJY = JEY = J(X\beta_1 + W\beta_2 + Z\beta_3) = \beta_1$

$\Rightarrow JX = I, JW = 0,\ \&\ JZ = 0.$

But $JZ = J(XH_1 + WH_2)$

$JZ = JXH_1 + JWH_2$

$JZ = H_1 + 0H_2 = H_1$

$\Rightarrow H_1$ must have zero, therefore $Z = WH_2$. But there are more linearly independent Z's than W's, which contradicts the assumption $N \leq \sum_{i=0}^{H} \binom{k}{i} + \binom{k-1}{l-1}$. □

THEOREM A .1 : Let $N = 2^{k-p}$, let H be the largest integer such that $N \geq \sum_{i=1}^{H} \binom{k}{i}$ and let I be the indicator function. Then

$$R_{max} \leq 1 + 2H + I\left[N \geq \sum_{i=0}^{H} \binom{k}{i} + \binom{k-1}{H}\right]. \qquad (A.2)$$

PROOF : Suppose $R_{max} = 1 + 2l$ is odd. Then the parameters estimated include the grand mean, the k main effects, the $\binom{k}{2}$ two-factor interaction, ..., and the $\binom{k}{l}$ l-factor interactions. This requires a minimum of $1+\binom{k}{1}+\binom{k}{2}+...+\binom{k}{l}=\Sigma_{i=0}^{l}\binom{k}{i}\leq N$ runs. Then $l\leq H$ and $R_{max}=1+2l\leq 1+2H$ so that $(A.2)$ holds. Now suppose $R_{max}=2l$ is even. From the definition of H we have $H\geq l-1$. If $H=l-1$, then $R_{max}=2l=2+2H$ and $(A.2)$ is satisfied since $N\geq\Sigma_{i=0}^{l-1}\binom{k}{i}+\binom{k-1}{l-1}$ by the preceding lemma. If $H\geq l$ then $R_{max}=2l\leq 2H$ and $(A.2)$ holds.

□

APPENDIX B

LEMMA B.1

$n(O)=2^{p-1}, n(E)=2^{p-1}-1$, where $n(O)$ & $n(E)$ are defined in (4.1).

PROOF

Consider a set $\phi=\{i_1,i_2,...i_s\}$, $0<i_1<i_2<\ .\ .\ .\ <i_s<p+1$. There are 2^p-1 such subsets. Let S denote the set of entire collection of these subsets.

Let s be an odd number. Clearly there are 2^{p-s} members of S which intersect ϕ in only i_j, $j=1,2,...s$ and no other members of ϕ. Similarly there are $\binom{s}{3}$ possible triplets $\{i_{j_1}i_{j_2}i_{j_3}\}\subset\phi$ and for each of these triplets there are exactly 2^{p-s} members of S which intersects ϕ in this triplet and no other members of ϕ. Continuing in a similar manner, for $i_{j_1},i_{j_2},...i_{j_s}\in\phi$ there are 2^{p-s} sets in S which intersect ϕ in $\{i_{j_1},i_{j_2},...,i_{j_s}\}$ and no other members of ϕ. The totality of these members give the number of symbols in O and is given as,

$$n(O)=\binom{s}{1}2^{p-s}+\binom{s}{3}2^{p-s}+...\binom{s}{s}2^{p-s}$$

$$= 2^{p-s}\left[\binom{s}{1} + \binom{s}{3} + ...\binom{s}{s}\right] \qquad (B.1)$$

We know the Binomial Series,

$$\left[\binom{s}{0} + \binom{s}{1} + ...\binom{s}{s}\right] = 2^s \qquad (B.2)$$

If s is an odd then, we have

$$\binom{s}{1} = \binom{s}{s-1}, \binom{s}{3} = \binom{s}{s-2}, ...\binom{s}{s} = \binom{s}{s-s} \qquad (B.3)$$

$(B.2)$ can be written as

$$\left[\binom{s}{1} + \binom{s}{3} + ...\binom{s}{s}\right] + \left[\binom{s}{0} + \binom{s}{2} + ...\binom{s}{s-1}\right] = 2^s$$

Then $(B.3) \Rightarrow$

$$2\left[\binom{s}{1} + \binom{s}{3} + ...\binom{s}{s}\right] = 2^s$$

$$\left[\binom{s}{1} + \binom{s}{3} + ...\binom{s}{s}\right] = 2^{s-1} \qquad (B.4)$$

substituting $(B.4)$ in $(B.1)$, we get

$$n(O) = 2^{p-s}\{2^{s-1}\} = 2^{p-1}.$$

Since Cardinality of S = Cardinality of E + Cardinality of O i.e. $O \bigcup E = S$ and $\#S = 2^p - 1$, we have

$$n(E) = (2^p - 1) - (2^{p-1}) = 2^{p-1} - 1$$

. $\square$