CHAPTER 4

MINIMUM ABERRATION CRITERION

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4.1 INTRODUCTION

In the previous chapter, we have discussed two methods for selecting a suitable fraction for a given design. There was clear-cut distinction between the set of interactions which are to be estimated (called the requirements set R) and not to be estimated. These both sets were assumed to be well defined before conducting the experiment. But in many situations such partition of the set of interactions is not possible due to lack of prior information. In such cases the selection of a good fraction should be based on a criterion called as Minimum Aberration Criterion(MAC).

In this chapter, we discuss a criterion of minimum aberration and the algorithm developed by Fries and Hunter (1980) for generating a best fraction of a design according to the minimum aberration criterion for $N = 2^{k-p}$ runs. It is observed that all designs with the highest possible resolution are not always good. In the next section, we define the minimum aberration criterion and illustrate it with the help of an example.

In Chapter-II, we have studied the criterion of resolution of a

FFD. A design with the highest possible resolution is always desired. But there may be many FFD's of the same highest resolution. So there arises a question, among these designs which is the best one? Fries and Hunter (1980) proposed the criterion of minimum aberration for distinguishing between designs of the same resolution. Later this criterion is considered by Franklin (1984), Chen and Wu (1991), Chen (1992), Tang and Wu (1996) and Suen (1997) among others. Minimum aberration criterion plays a fundamental role in the choice of FFD.

In Section 4.2, we discuss the concept of a minimum aberration and illustrate it with an example. Section 4.3 deals with necessary and sufficient conditions for the existence of a defining relation with specified word length pattern. At the end in Section 4.4 we present an algorithm suggested by Fries and Hunter (1980) for selecting a minimum aberration design from among all possible 2^{k-p} FFD's with the same maximum resolution. The algorithm is illustrated with the help of examples.

4.2 MINIMUM ABERRATION CRITERION

4.2.1 THE CONCEPT OF MINIMUM ABERRATION

In the following we use a simple example to explain the motivation for the minimum aberration criterion.

Consider an example, a 2^{7-2} design with resolution IV. There are

7 factors each at two levels and requires 32 runs. Consider the following three designs each of resolution IV with respective defining relations,

$$D_1: I = ABCF = BCDG = ADFG$$
$$D_2: I = ABCF = ADEG = BCDEFG$$
$$D_3: I = ABCDF = ABCEG = DEFG$$

There arises a question, from these three designs which design is best . In designs of resolution IV, the main effects are not aliased with each other and two- factor interactions, but the two-factor interactions are aliased with each other. Thus, unconfounded estimates are obtained for all the main effects if one assumes that the three factor and higher order interactions are negligible. If we make this assumption, then these designs are to be compared with regard to confounding among two-factor interactions. For these designs, the alias sets which include two-factor interactions aliased with each other are given below,

$D_1: AB + CF,\ \ AC + BF,\ \ AD + FG,\ \ AG + DF,\ \ BD + CG,$

$BG + CD,\ \ AF + BC + DG.$

$D_2: AB+CF,\ \ AC+BF\ \ AD+EG,\ \ AE+DG,\ \ AF+BC,\ \ AG+DE.$

$D_3: DE + FG,\ \ DF + EG,\ \ DG + EF.$

It is seen that among the two-factor interactions, there is greatest amount of confounding in design D_1 and the least in D_3. Therefore, design D_3 can be treated as the best of the three designs.

Note that the extent of confounding among two factor interactions is related with the word length pattern. In the above example, the word length pattern (defined in Section (2.4)) for the three designs is (4,4,4) , (4,4,6) and (4,5,5) respectively. The smallest word length in all the three designs is 4. However, D_3 has only one word of length 4, where as D_2 has two and D_1 has three. Thus design D_3 contains the smallest number of words of the smallest length i.e. four among the three designs, consequently, it has minimum number of two factor interactions aliased with each other. Such a design is called as minimum aberration design. A formal definition of minimum aberration design follows:

DEFINITION 4.1

For a design d, let $A_r(d)$ be the number of words of length r in the defining relation. For any two 2^{k-p} fractional factorial designs d_1 and d_2, if r is the smallest positive integer (> 2) such that $A_r(d_1) \neq A_r(d_2)$, then d_1 is said to have less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. A design d has minimum aberration if there is no other design with less aberration. □

Thus, to compare two designs using the resolution criterion, one considers the shortest word length in each defining relation. If the two designs have the same shortest word length, then these are regarded as being equivalent, according to the criterion of resolution. On the otherhand while comparing designs using aberration criterion, one continues to compare the wordlengths in the defining relation in ascending order until one design is ranked superior than the other.

Note that minimum aberration automatically implies maximum resolution of a design. Thus, when there are two or more designs of the same resolution R, the minimum aberration criterion takes a design with fewer words of the minimum length. In a resolution R design, the main effects are aliased with interactions of order $R-1$, the two-factor interactions are aliased with interactions of order $R-2$ and so on. If we have given a design with maximum resolution R and minimum aberration, this implies that a design has the minimum number of words of length R which means that the smallest number of main effects will be aliased with interactions of order $R-1$, the smallest number of two-factor interactions will be aliased with interactions of order $R-2$ and so on. Hence the concept of minimum aberration is a natural extension of resolution.

The example presented next illustrates these points.

4.2.2 ILLUSTRATION

Consider the following 2^{7-2} FFD's d_1 *and* d_2 of resolution IV but have different word length patterns.

$$d_1 : I = DEFG = ABCDF = ABCEG$$
$$d_2 : I = ABCF = ADEG = BCDEFG$$

The word length patterns are (recall that $W(d) = (A_3, A_4, \ldots)$ where $A_i = \#$ words of length i)

$$W(d_1) = (0 \ \ 1 \ \ 2 \ \ 0 \ \ 0), \qquad W(d_2) = (0 \ \ 2 \ \ 0 \ \ 1 \ \ 0)$$

The first nonzero number in $W(d_1)$ represents the smallest word length in the defining relation. Here, A_4's are nonzero, $A_4(d_1) = 1$

and $A_4(d_2) = 2$ and $A_4(d_1) < A_4(d_2)$, hence d_1 has less aberration than d_2. Consequently in d_1 the amount of confounding among two factor interactions is less than that in d_2. In design d_1, there are three pairs of two-factor interactions (2fi's) aliased with each other, $DE + FG, \quad DG + EF, \quad DF + EG$ while design d_2 has six pairs of two factor interactions (2fi's) aliased with each other, $AB + CF, \quad AC + BF, \quad AD + EG, \quad AE + DG, \quad AF + BC, \quad AG + DE$.

Usually all three factor interactions and higher order interactions are assumed to be negligible.

The algorithm by Fries and Hunter (1980) for obtaining a minimum aberration design is discussed in the section 4.4. For the development of this algorithm certain necessary and sufficient conditions for the existence of a defining relation with a specified word length pattern are required. These conditions are discussed in the next section.

4.3 SOME NECESSARY AND SUFFICIENT CONDITIONS

4.3.1. NOTATIONS AND PRELIMINARIES

Let $i_1, i_2, \ldots i_s$ be s integers such that $0 < i_1 < i_2 < \ldots < i_s < p + 1$. The i^{th} generator in a defining relation is denoted by $W(i)$ and the generalized interaction of $i_1^{th}, i_2^{th}, \ldots$ and i_s^{th} generator by $W(i_1, i_2, \ldots, i_s)$. There are exactly $2^p - 1$ words $W(i_1, i_2, ..., i_s)$ corresponding to the $2^p - 1$ symbols (unordered tuples) $(i_1, i_2, \ldots i_s)$. The length of the word $W(i_1, i_2, \ldots i_s)$ is denoted by $w = w(i_1, i_2, \ldots i_s)$. Let S denote the set of all $2^p - 1$ symbols $(i_1, i_2, \ldots i_s)$ and Let $O = O(i_1, i_2, \ldots i_s)$ ($E = E(i_1, i_2, \ldots i_s)$) denote the class of symbols from S which contain an odd number of indices (none or an even number of indices) from $(i_1, i_2, \ldots i_s)$. For example, let $p = 3$ and $2^p - 1 = 7$. Then, $S = \{(1),\ (2),\ (3),\ (12),\ (13),\ (23),\ (123)\}$. Here for example, $O\{(13)\}$ is the set of all symbols containing an odd number of indices of from (1,3) i.e., either (1) or (3) but not (1 3) both. Thus $O\{13\} = \{(1)\ \ (3)\ \ (12)\ \ (23)\}$. and $E\{13\} = O\{13\}^c = \{(2)\ \ (13)\ \ (123)\}$. Let $n(O)$ and $n(E)$ denote the cardinality of O and E respectively. In Lemma $B.1$ of Appendix B it is proved that

$$n(O) = 2^{p-1}, \quad n(E) = 2^{p-1} - 1 \tag{4.1}$$

DEFINITION 4.2 :

The symbol $t(i_1, i_2, \ldots i_s)$ is defined to be the number of letters which appear in all of the s generators $W(i_1), W(i_2), \ldots W(i_s)$ but in no other generators. □

For example, $t(i_1)$ denotes the number of letters appearing only in the i_1^{th} generator, $t(i_1, i_2)$ denotes the number of letters appearing in only i_1^{th} and i_2^{th} generators and so on. Consider an example of a 2^{6-3} fraction i.e. $k = 6$ and $p = 3$. The three generators be $W(1) = BCD$, $W(2) = ACDEF$, $W(3) = ACF$ respectively. Consider Figure 4.1 to determine the set of t's defined above.

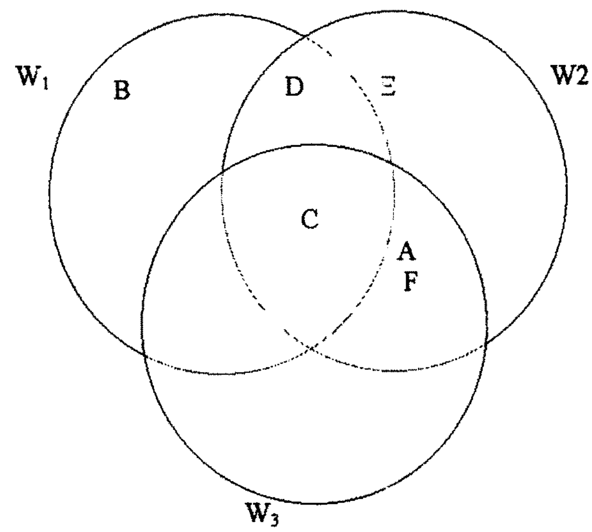


FIGURE 4.1

In this example, according to the definition 4.2 the t's are given by

$$\left.\begin{array}{llll} t(1) = 1 & t(2) = 1 & t(3) = 0 & \\ t(12) = 1 & t(13) = 0 & t(23) = 2 & t(123) = 1 \end{array}\right\} \qquad (4.2)$$

Next we discuss some necessary and some sufficient conditions for the existence of a defining relation with a given word length pattern.

4.3.2 CONSTRUCTION OF A DEFINING RELATION WITH A GIVEN SET OF t's

In this subsection we discuss conditions for the existence of a defining relation with a given word length pattern which are derived by Burton and Connor (1957). The main result is discussed in Theorem 4.2. We also illustrate the construction of a defining relation from a given word length pattern. It is proved that the knowledge of the t's is sufficient for the construction of a defining relation. This defining relation is unique apart from the renaming of letters (factors).

Let $S(i_1, i_2, \ldots i_s)$ be the set of letters which appear only in $W(i_1), W(i_2), \ldots W(i_s)$ and no others. Note that, the sets $S(i_1, i_2, \ldots i_s)$ are disjoint and form a partition of $\bigcup_i^m W_i$. (illustrated in the above diagram) and $t(i_1, i_2, \ldots i_s)$ is the cardinality of the set $S(i_1, i_2, \ldots i_s)$. Since under the assumption made, each of the k letters appears in at least one of the words, we must have,

$$\sum_S t(i_1, i_2, \ldots i_s) = k \tag{4.3}$$

where $\sum_S$ denotes sum over $(i_1, i_2, ..., i_s) \in S$. Also, for the same reasons, any set of t's which are positive integers or zero, and satisfy (4.3) corresponds to a constructible defining relation involving k factors, apart from the relabeling of the factors by assigning $t(i_1, i_2, \ldots i_s)$ distinct factors to the set $S(i_1, i_2, \ldots i_s)$ with no two sets receiving any common letter. This gives $W(i) = \bigcup_{(i_1 \ldots i_s) \supset i} S(i_1, i_2, \ldots i_s)$, $i = 1, 2, ..., p$. For example, given the values of t's as in (4.2), the set of W's can be determined with the help of Figure (4.2) displayed on the next page.

We assign $t(1) = 1$ factor say C to $S(1)$, $t(2) = 1$ factor say E to $S(2)$, $t(12) = 1$ factor say D to $S(12)$, $t(23) = 2$ factors say A and F to $S(23)$, and $t(123) = 1$ factor say B to $S(123)$. This assignment gives the generators, $W(1) = S(1) \cup S(12) \cup S(123) = BCD$, similarly $W(2) = ABDEF$, and $W(3) = ABF$ are obtained and the defining relation for the above design is then given by,

$I = BCD = ABDEF = ABF = ACEF = ACDF = DE = BCE$

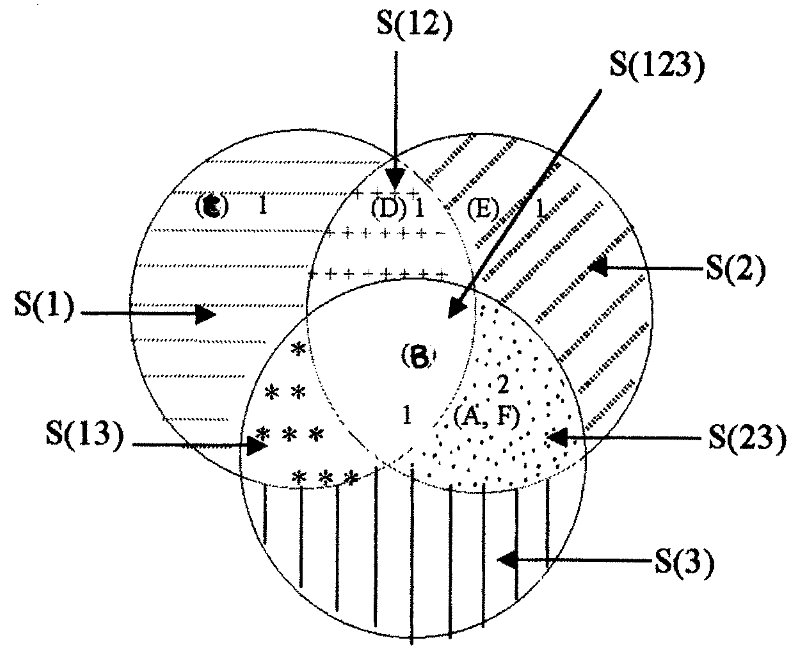


FIGURE 4.2

The above discussion can be rephrased in the following lemma.

<u>**LEMMA**</u>**4.1**

If a given set of non-negative integers $t(i_1, i_2, ..., i_s)$ corresponding to each $(i_1, i_2, ..., i_s) \in S$, satisfies (4.3) then a defining relation can be constructed with these values of t's. Conversly, if a defining relation exists, then the corresponding t's satisfy (4.3). □

Before presenting the main results, we prove some lemmas which are useful in the proofs of main results.

4.3.3 SOME PRELIMINARY LEMMAS

In this section, we present some lemmas that are required in the proof of main result. First we state the following necessary conditions for the existence of a defining relation with a specified word length pattern, (without proof) (cf. Burton and Connor (1957)) that are required in the proof of Lemma 4.3 and Theorem 4.2 and also in the algorithm suggested by Fries and Hunter (1980).

Brownlee, Kelly, and Loraine (1948) have obtained the following necessary conditions for the existence of the defining relation with a specified word length pattern. We state these conditions without proof. Let $w_1, w_2, \ldots w_m$, $m = 2^p - 1$, denote the lengths of words in a defining relation for a 2^{k-p} fraction consisting of m words. Then the w's must satisfy the following conditions :

i) $\sum_{i=1}^{m} w_i = 2^{p-1}k$ (4.4)

ii) Either the w's all are even or exactly 2^{p-1} of them are odd. (4.5)

iii) If some $w = k$, the remaining w's must be divisible into pairs such that the total of each pair is k.

iv) If some $w = 1$, the remaining numbers must be divisible into pairs such that the numbers in each pair differ by 1. □

Note that, these conditions do not require that the given word lengths of w's are associated with particular generators or generalized interactions.

Next we present some lemmas which are used in the proof of Theorem 4.2.

LEMMA 4.2 :

For a 2^{k-p} fractional factorial design, for every $(i_1, i_2, ..., i_s) \in S$, the t's defined in Definition (4.2) must satisfy the equations

$$\sum_{O(i_1,i_2,...i_s)} t(j_1, j_2, . \quad . \quad . \quad j_r) = w(i_1, i_2, . \quad . \quad . \quad i_s) \tag{4.6}$$

Further, these $2^p - 1$ equations uniquely determine the w's from the t's apart from the relabeling of the letters.

PROOF :

Note that, for any fixed symbols $(i_1, i_2, ..., i_s) \in S$ the generalized interaction $W(i_1 i_2 ... i_s)$ of the words $W(i_1), W(i_2), ...W(i_s)$, is nothing but the product modulo 2 of all these generators. In carrying out the product, if a particular letter appears in an even number of generators among $W(i_1), W(i_2), ...W(i_s)$, then in the product its exponent will be an even number which is zero mod 2 and hence this letter will not appear in the product. Similarly, if a letter appears in an odd number of W's among $W(i_1), W(i_2), ...W(i_s)$, its exponent will be an odd number which is one mod 2 and hence it will appear in the product with exponent one. Thus the generalized interaction $W(i_1 i_2 ... i_s)$ of $W(i_1), W(i_2), ...W(i_s)$ is nothing but the string of letters which appears an odd number of times among $W(i_1), W(i_2), ...W(i_s)$. Noting the definition of t's, this gives,

$$\sum_{O(i_1,i_2,...i_s)} t(j_1, j_2, . \quad . \quad . \quad j_r) = w(i_1, i_2, . \quad . \quad . \quad i_s)$$

There are $2^p - 1$ such equations corresponding to the $2^p - 1$ symbols $(i_1, i_2, ...i_s)$. Thus knowing t's , w's can be uniquely determined from these equations. □

We illustrate this with the following example.

EXAMPLE 1 : Consider a 2^{7-3} fractional design i.e. $k = 7$ *and* $p = 3$, suppose the t's are given by,

$$\text{t}(1) = 1 \quad \text{t}(2) = 1 \quad \text{t}(3) = 1$$
$$\text{t}(12) = 1 \quad \text{t}(13) = 1 \quad \text{t}(23) = 1 \quad \text{t}(123) = 1$$

Note that t's satisfy (4.3) i.e.$\sum_S t = 7$. This is represented diagrammatically in Figure(4.3) on next page.

Here $S = \{(1), (2), (3), (12), (13), (23), (123)\}$, and $O(1) = \{(1), (12), (13), (123)\}$, $E(1) = \{(2), (3), (23)\}$

From the equation (4.6), we get,

$\sum_{O(1)} t(j_1 j_2 ... j_r) = w(1)$ i.e $w(1) = t(1) + t(12) + t(13) + t(123) = 4$

Other w's can be similarly determined and are given by $w(2) = 5$ $w(3) = 5$ $w(12) = 4$ $w(13) = 4$ $w(23) = 4$ $w(123) = 4$ □

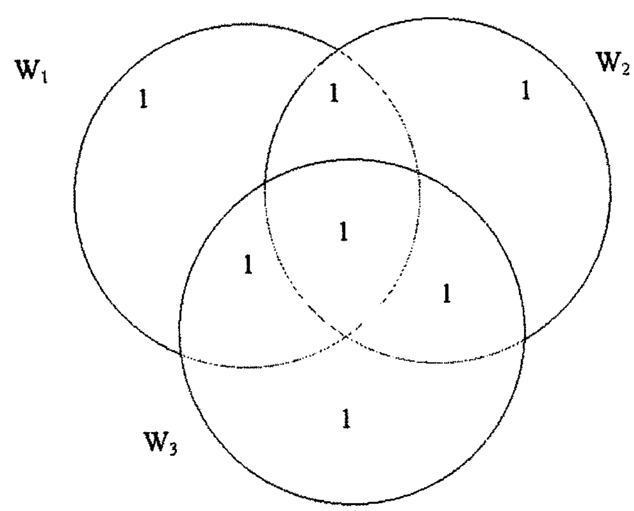


FIGURE 4.3

LEMMA 4.3

For a 2^{k-p} fractional design, let $w(j_1, j_2, ..., j_r)$, $(j_1, j_2, ..., j_r) \in S$ be a given set of non-negative integers. Then this set of w's uniquely determine the t's defined in definition 4.2 . Further, these t's satisfy the $2^p - 1$ equations (corresponding to each $(i_1, i_2, ..., i_s) \in S$)

$$\sum_O w(j_1, j_2, . . . j_r) - \sum_E w(j_1, j_2, . . . j_r) = 2^{p-1} t(i_1, i_2, . . . i_s) \qquad (4.7)$$

where $O = O(i_1, i_2, ..., i_s)$, $E = E(i_1, i_2, ..., i_s)$.

PROOF

Consider a set of w's, $w(j_1, j_2, ..., j_r)$, $(j_1, j_2, ..., j_r) \in S$. Let the t's be as defined in definition (4.2) which must satisfy (4.3). There are $2^p - 1$ t's corresponding to each $(i_1, i_2, ..., i_s) \in S$. Let $t(0)$ denote a dummy variable which is identically equal to zero. Then adding $t(0)$ to the LHS of the equations (4.3), we get,

$$t(0) + \sum_S t(j_1, j_2, \ldots j_r) = k \tag{4.8}$$

where $\sum_S$ denotes sum over $(j_1, j_2, ..., j_r) \in S$. For fixed $(i_1, i_2, ..., i_s) \in S$, multiplying equation (4.6) by 2, subtracting equation (4.8) from it, and noting that $\sum_S = \sum_O + \sum_E$, we get,

$$2\sum_C t(j_1, j_2, ...j_r) - t(0) - \left[\sum_O t(j_1, j_2, ..., j_r) + \sum_E t(j_1, j_2, ...j_r)\right] = 2w(i_1 i_2 ... i_s) - k$$

$$\Rightarrow -t(0) - \sum_E t(j_1, j_2, ..., j_r) + \sum_O t(j_1, j_2, ...j_r) = 2w(i_1 i_2 ... i_s) - k \tag{4.9}$$

where $O = O(i_1, i_2, ...i_s), E = E(i_1, i_2, ...i_s)$ are as defined in section 4.3.1.

Note that, corresponding to each $(i_1, i_2, ..., i_s) \in S$ we get an equation and there are $2^p - 1$ members of S, so that (4.9) is a set of $2^p - 1$ such equations. This together with (4.8) gives 2^p equations and in each equation there are 2^p terms. Writting this in the matrix form we get,

$$H\underline{t} = \underline{b} \tag{4.10}$$

where H is the matrix of coefficients in equations (4.8) and (4.9) of order $2^p \times 2^p$. It can be shown that the matrix H is a Hadamard matrix (that is a matrix whose elements are +1 or -1, whose columns are orthogonal to each other). Let $\underline{t}$ denotes the column vector $\left[t(0),\ t(1),\ \ldots,\ t(12...p)\right]'_{2^p \times 1}$

and $\underline{b}$ denotes the column vector $\left[k, 2w(1)-k, \dots\ 2w(12...p)-k\right]'_{2^p\times 1}$.
Multiplying equations (4.10) by $2^{-p/2}$, the matrix of coefficients $2^{-p/2}H = C$ becomes an orthogonal matrix. Then (4.10) is equivalent to

$$C\underline{t} = \underline{x} \tag{4.11}$$

where $\underline{x} = 2^{-p/2}\underline{b}$. Since C is orthogonal, $C^{-1} = C'$. This gives, $\underline{t} = C'\underline{x}$ where $C' = 2^{-p/2}H'$.
$\Rightarrow \underline{t} = (2^{-p/2}H')(2^{-p/2}\underline{b})$
$\Rightarrow \underline{t} = 2^{-p}H'\underline{b}$
This gives,

$$2^p\underline{t} = H'\underline{b} \tag{4.12}$$

Note that, the rows and columns of the matrix H can be labeled by the sets of the type $(i_1i_2...i_s)$. In (4.12), the right hand side of the equation corresponding to $t(i_1i_2...i_s)$ is the product of $(i_1i_2...i_s)^{th}$ row of H' i.e. $(i_1i_2...i_s)^{th}$ column of H with the column vector $\underline{b}$. The elements of H corresponding to the row $(i_1i_2...i_s)$ and column $(j_1j_2...j_r)$ will be $+1$ if the set $(j_1j_2...j_r)$ has an odd number of letters common with the set $(i_1i_2...i_s)$ and -1 otherwise. Also from (4.1), we have $n(O) = 2^{p-1}$ and $n(E) = 2^{p-1}-1$ number of letters in O and E. Noting these observations, we get

$$2^pt(i_1i_2...i_s) = k + 2\left[\sum_O w(j_1, j_2, ..., j_r) - \sum_E w(j_1, j_2, ... j_r)\right]$$

$$+k\Big[(2^{p-1}-1)-2^{p-1}\Big]$$

$$2^p t(i_1 i_2 ... i_s) = 2\Big[\sum_O w(j_1, j_2, ..., j_r) - \sum_E w(j_1, j_2, ... j_r)\Big]$$

$$\Rightarrow 2^{p-1} t(i_1, i_2, ... i_s) = \sum_O w(j_1, j_2, \ldots j_r) - \sum_E w(j_1, j_2, \ldots j_r)$$

as required.

These equations uniquely determine the t's from the w's (since the coefficient matrix H is non-singular). □

A sufficient condition for the existence of a defining relation with a specified word length pattern is given in the next theorem.

4.3.4. MAIN RESULTS : A NECESSARY AND SUFFICIENT CONDITION

THEOREM 4.1

For a given set of non-negative integers $w = w(j_1, j_2, ..., j_r)$, $(j_1, j_2, ..., j_r) \in S$. If the t's determined by (4.7) are non-negative integers such that $\sum_S t = k$ then there exists a defining relation with the word length pattern $w(j_1, j_2, ..., j_r)$, $(j_1, j_2, ..., j_r) \in S$.

PROOF

Consider a set of w's, $w(j_1, j_2, ..., j_r)$, $(j_1, j_2, ..., j_r) \in S$ and the t's be obtained using (4.7). If these t's are non-negative integers and satisfy $\sum_S t = k$ then a defining relation can be constructed with this set of t's as discussed in Lemma 4.1. This defining relation is unique apart from the relabeling of letters (factors). □

Next theorem gives a necessary condition for the existence of a defining relation with a given word length pattern.

THEOREM 4.2

For a 2^{k-p} fractional design, a necessary condition for the existence of a defining relation with given word length pattern $w(i_1, i_2, ..., i_s)$, $(i_1, i_2, ..., i_s) \in S$ is that there are positive integers $t(i_1, i_2, ..., i_s)$, $(i_1, i_2, ..., i_s) \in S$, among which at most k are positive whose sum is k and whose squares add to $2^{-p+2}\sum w^2 - k^2$, that is ,

$$\sum w^2 = 2^{p-2}(\sum t^2 + k^2) \tag{4.13}$$

here $w = w(i_1, i_2, ..., i_s)$, where $\sum$ denotes over $(i_1, i_2, ..., i_s) \in S$.

PROOF

From (4.11), we have

$$C\underline{t} = \underline{x}$$

where $C = 2^{-p/2}H$.

Then consider,

$$x'x = \underline{t}'C'C\underline{t}$$

Since C is an orthogonal matrix, $C^{-1} = C'$, this gives ,

$$\underline{x}'\underline{x} = \underline{t}'\underline{t} \quad \text{that is, } 2^{-p}\underline{b}'\underline{b} = \underline{t}'\underline{t} \qquad (\text{ since } \underline{x} = 2^{-p/2}\underline{b})$$

$$\Rightarrow 2^{-p}\left[k^2 + \sum_i^m \left(2w - k\right)^2\right] = \sum_S t^2$$

where $m = 2^p - 1$.

$$\Rightarrow k^2 + \sum_i^m \left(4w^2 + k^2 - 4wk\right) = 2^p \sum_S t^2$$

$$\Rightarrow k^2 + 2^2 \sum_i^m w^2 + k^2\left(2^p - 1\right) - 2^2 k \sum_i^m w = 2^p \sum_S t^2$$

$$\Rightarrow 2^2 \sum_i^m w^2 + 2^p k^2 - 2^2 k\left(2^{p-1}k\right) = 2^p \sum_S t^2 \qquad (\text{since} \sum_i^m w = 2^{p-1}k)$$

$$\Rightarrow 2^2 \sum_i^m w^2 + 2^p k^2 (1-2) = 2^p \sum_S t^2$$

$$\Rightarrow 2^2 \sum_i^m w^2 - 2^p k^2 = 2^p \sum_S t^2$$

$$\Rightarrow 2^2 \sum_i^m w^2 = 2^p \left(\sum_S t^2 + k^2\right)$$

$$\Rightarrow \sum w^2 = 2^{p-2}(\sum t^2 + k^2)$$

as required. □

Note that the lengths of the particular generators and their generalized interactions are not specified.

The next section deals with an algorithm for selecting a best fraction which has minimum aberration.

4.4 ALGORITHM SUGGESTED BY FRIES AND HUNTER(1980)

In this section, we discuss a method for selecting a best (having minimum aberration) fraction from the set of all possible 2^{k-p} fractional factorial design of highest resolution. This method is suggested by Fries and Hunter (1980). First we present the algorithm and then illustrate it by an example. Here, we assume that the maximum resolution R_{max} to be known.

The algorithm first searches a possible best word length pattern according to a minimum aberration criterion for a value of R_{max} given by (2.3), (2.4) and (2.5) discussed in section 2.4. Then a set of t's defined in Definition (4.2) is the t's corresponding to these w's are retrieved, and examined for the conditions (4.3) and (4.13). If t's satisfy these conditions then a corresponding defining relation is obtained using lemma 4.1. If no set of t's is found , the algorithm is repeated with a smaller value of R_{max}. Next we present the algorithm.

4.4.1 ALGORITHM

Step 1 : Initially decide the value of R_{max} for given k and p which is obtained by using the bounds given in equations (2.3), (2.4) and (2.5).

Step 2 : Choose a best word length pattern (that is, according to minimum aberration criterion) with resolution $R_{max} = B$ such that $w_i \geq R_{max}$ that is words with lengths which are greater than R_{max} and satisfying $\sum_{i=1}^{m} w_i = 2^{p-1}k$ (cf. Section 4.3.1 condition (4.4)). If such word length pattern does not exist, then set a new word length pattern with resolution $B = oldB - 1$, continue in this manner untill the best word length pattern with resolution B is found.

Step 3 : Check the number of odd and even words in the word length pattern either all w's should be even or exactly 2^{p-1} of them should be odd. If the chosen word length pattern does not satisfy this condition, then discard that word length pattern and move to step 2. (cf. Section 4.3.1, condition (4.4) and (4.5))

Step 4 : Find a (another) possible set of t-values (possibly more than one) satisfying equations (4.3) and (4.13). If no set of t - values exists but a minimum aberration design has been found, then stop the process. Otherwise eliminate the current word length pattern from consideration and move to step 2. (cf. Lemma 4.1, Theorem 4.2)

Step 5 : Compute the lengths of generators and their generalized interactions using (4.6). Find the generators up to a relabeling of factors. This gives a desired minimum aberration design. (cf. Lemma 4.2)

Step 6 : Decide the next nonisomorphic (that is, sets of t's which are merely renaming of others not considered) assignment of t's to specific generators and their generalized interactions. If none exists, then go to step 4.

The algorithm is illustrated with the help of the following examples.

4.4.2 ILLUSTRATIONS

EXAMPLE 1 :

Consider a 2^{6-2} fraction with $k = 6$, $p = 2$. From Step 1 and Step 2, here $R_{max} = 2k/3 = 4$. Therefore $w_i \geq 4$ and the wordlength pattern is , $W = \{w_1 \ \ w_2 \ \ w_{12}\}$. Here $\sum_{i=1}^{m} w_i = 2^{p-1}k = 12$, where $m = 2^p - 1 = 3$.

According to Step 3, the word length patterns satisfying condition (4.4) and (4.5) as discussed in section (4.3) is $W_1 = (4 \ \ 4 \ \ 4)$. W_1 satisfies both conditions, that is the word length pattern W_1 has all words of even lengths so that it satisfies (4.5). There does not exist any other word length pattern with resolution IV. Therefore W_1 is the best word length pattern.

Now we come to Step 4, from equation (4.3), we have $\sum_S t(i_1 i_2 \ldots i_s) = k \Rightarrow \sum t = 6$ and from equation (4.13), we have $\sum t^2 = 12$. The only assignment that satisfies these conditions is 2 , 2 and 2. The assignment of t's are $t(1) = 2, \quad t(2) = 2, \quad t(12) = 2$.

Further from Step 5, employing (4.6), the lengths of the generators and their generalized interactions are given as $w(1) = t(1) + t(12)$ $= 4$, similarly $w(2) = 4$ and $w(12) = 4$. Therefore for given values of t's, the set of w's can be determined with the help of the following Figure 4.4 displayed on the next page.

We assign $t(1) = 2$ factor, say A, B to $S(1)$, $t(2) = 2$, say factors C, D to $S(2)$, $t(12) = 2$ factors, say E, F to $S(12)$. This assignment gives the generators, $W(1) = S(1) \cup S(12) = ABEF$ similarly $W(2) = CDEF, \quad W(12) = ABCD$. The defining relation for the above design is then given by,

$$I = ABEF = CDEF = ABCD.$$

Equivalently, the assignment of 2 factors each to the three disjoint sets $S(1), S(2)$ and $S(12)$ can be made in totally 120 ways each giving rise to a desired design.

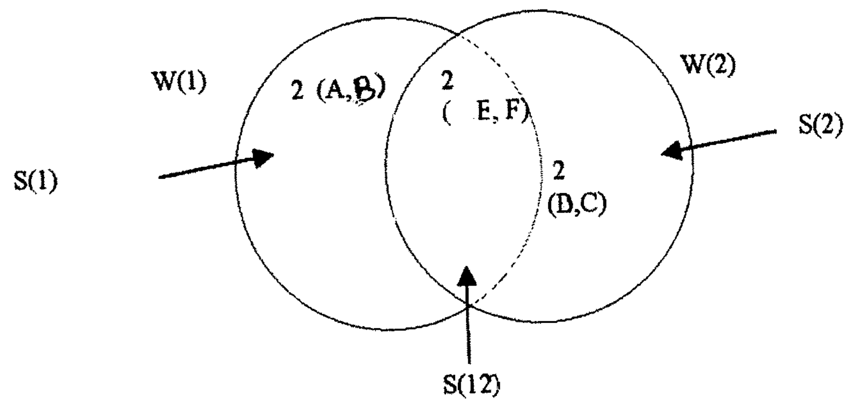


FIGURE 4.4

EXAMPLE 2 :

Consider a 2^{6-3} fraction design with $k = 6$, $p = 3$. Step 1 and Step 2 gives, $R_{max}(\text{B}) = 2k/3 = 4$ (as discussed in section 2.4.1 in equation (2.3)). Therefore $w_i \geq 4$, the wordlength pattern is , $W = \{w_1 \;\; w_2 \;\; w_3 \;\; w_{12} \;\; w_{13} \;\; w_{23} \;\; w_{123}\}$ such that $\sum_{i=1}^{m} w_i = 2^{p-1}k = 24$, where $m = 2^p - 1 = 7$. Here for a resolution IV design, since $w_i \geq 4 \;\forall\; i$, $\sum_{i=1}^{m} w_i \geq 28$ therefore the word length pattern satisfying $\sum w = 24$ with resolution $R_{max} = 4$ does not exist. Therefore setting a new resolution $R_{max} = oldB - 1 = 3$ and $\sum w = 24$, a possible word length pattern is , $W = \{3 \;\; 3 \;\; 3 \;\; 3 \;\; 4 \;\; 4 \;\; 4\}$. Here there are exactly four words having odd length so that condition (4.5) satisfies. Next we come to Step 4.

From equation (4.3), we have $\sum_{S} t(i_1 i_2 ... i_s) = k \Rightarrow \sum t = 6$ and from equation (4.13), we have $\sum t^2 = 6$, from which it follows that

t's must take the values 1, 1, 1, 1, 1, 1. The assignment of t's are $t(1) = 1$, $t(2) = 1$, $t(3) = 1$ $t(12) = 1$, $t(13) = 1$, $t(23) = 1$ and $t(123) = 0$.

From Step 5, employing (4.6), the lengths of the generators and their generalized interactions are given as $w(1) = t(1) + t(12) + t(13)$ $= 3$, similarly $w(2) = 3$, $w(3) = 3$, $w(12) = 4$, $w(13) = 4$, $w(23) = 4$, and $w(123) = 3$. Therefore, for given values of t's, the set of w's can be determined with the help of the following Figure 4.5 displayed on the next page.

We assign one factor each to each disjoint set , $S(1), S(2), S(12)$, $S(13), S(23)$ and no other to $S(123)$. This assignment gives the generators, $W(1) = S(1) \cup S(12) \cup S(13) = ACE$ similarly $W(2) = BCF$, $W(3) = DEF$, and the defining relation for the above design is then given by,

$$I = ACE = BCF = DEF = ABEF = ACDF = BCDE = ABD.$$

Similarly other assignments assigning one factor each to six disjoint sets and keeping one set empty yields the following defining relations.

$$\begin{array}{ccccccccccccccc}
I & = & ABE & = & BCDE & = & DEF & = & ACD & = & ABDF & = & BCF & = & ACEF \\
I & = & ABDE & = & BCD & = & DEF & = & ACE & = & BCEF & = & ABF & = & ACDF \\
I & = & ABF & = & BCDF & = & DEF & = & ACD & = & BCE & = & ABDE & = & ACEF \\
I & = & ADF & = & ABCD & = & CDEF & = & BCF & = & ACE & = & ABEF & = & BDE \\
I & = & ABCD & = & BCE & = & CDEF & = & ADE & = & ABEF & = & BDF & = & ACF \\
I & = & ABEF & = & BCDE & = & DEF & = & ACDF & = & ABD & = & BCF & = & ACE
\end{array}$$

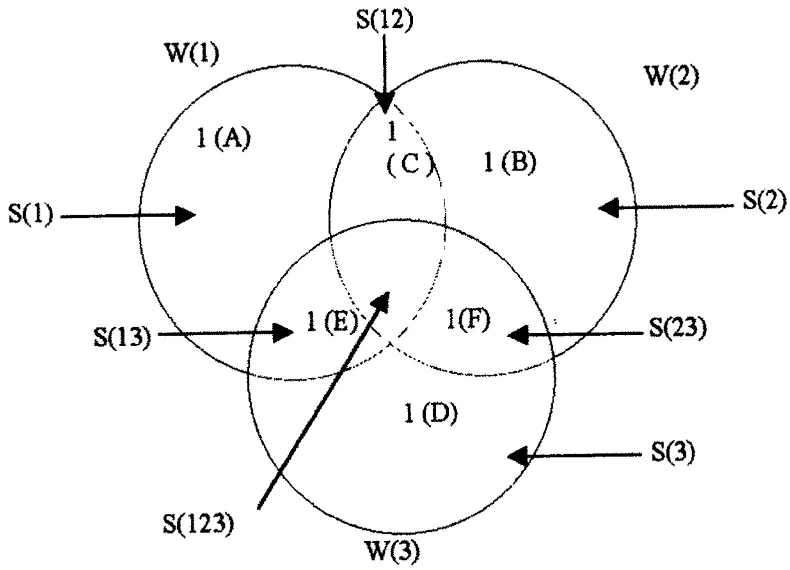
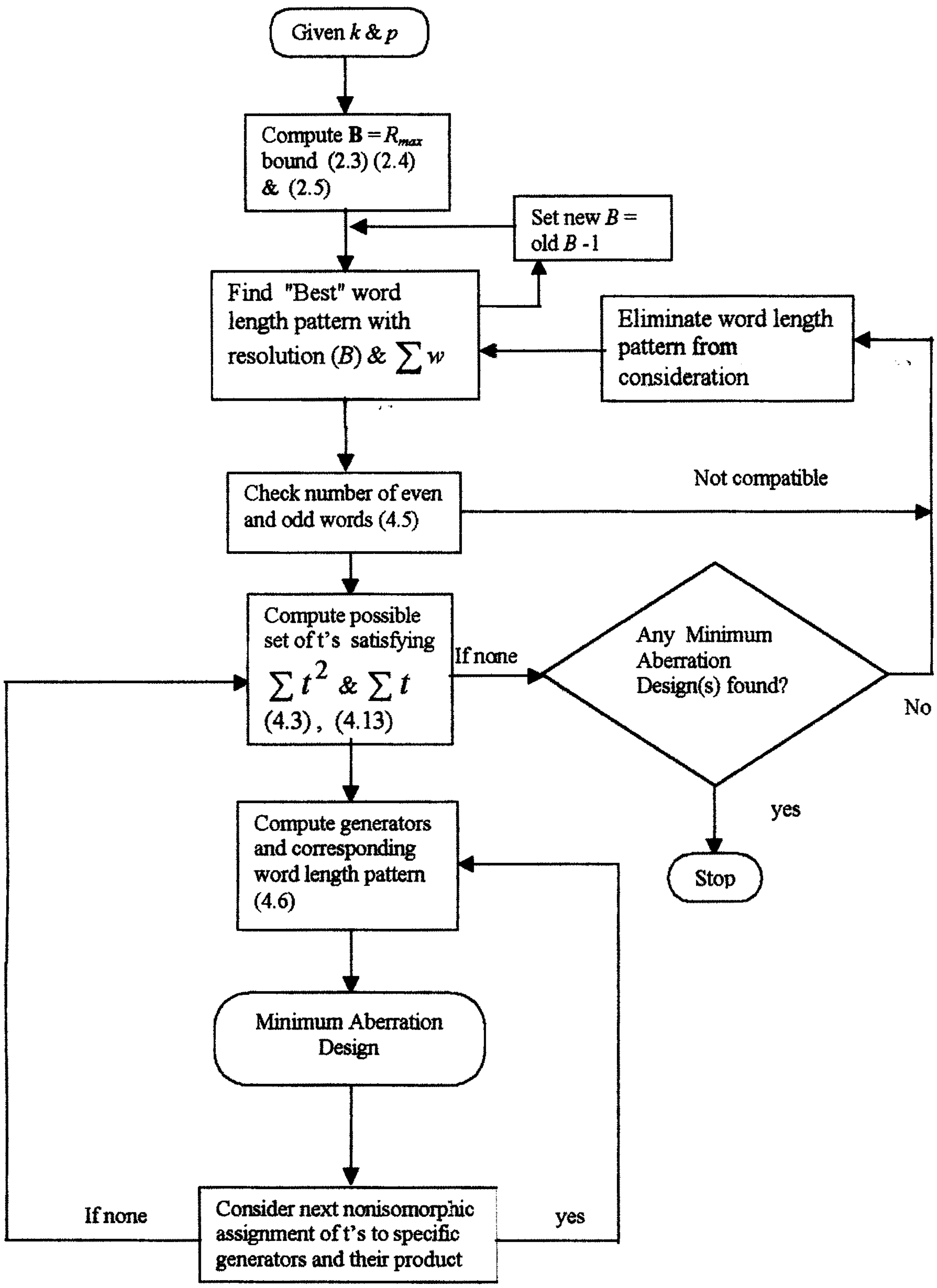


FIGURE 4.5

In the next Chapter, we focus on some other aspects of minimum aberration criterion for selecting a best fraction.

FLOW CHART



Given k & p
Compute $\mathbf{B} = R_{max}$ bound (2.3) (2.4) & (2.5)
Set new B = old B - 1
Find "Best" word length pattern with resolution (B) & $\sum w$
Eliminate word length pattern from consideration
Check number of even and odd words (4.5)
Not compatible
Compute possible set of t's satisfying $\sum t^2$ & $\sum t$ (4.3), (4.13)
If none
Any Minimum Aberration Design(s) found?
No
yes
Stop
Compute generators and corresponding word length pattern (4.6)
Minimum Aberration Design
If none
Consider next nonisomorphic assignment of t's to specific generators and their product
yes