CHAPTER - V

INEQUALITIES FOR PROBABILITY CONTENTS OF CONVEX SETS VIA GEOMETRIC AVERAGE

This chapter is based on a paper by Moshe Shaked and Y.L. Tong [13]. It establishes an inequality in probability contents among sets ordered by majorization at the same time comparable through their geometric averages. In other words it is a technique of locating the exact parametric values of a set with fixed shape and volume that would maximize the probability content. In specific it deals with rectangles and ellipsoids of fixed volume.

5.A. Conceptual Background :

Consider n random variables. Call them $X_1, X_2, \ldots, X_n$. Let f be the joint p.d.f. which is absolutely continuous, with respect to Lebsegue's measure. Define the set of points

$$A(a) = \left\{ x : |x_i| < a_i;\ i = 1, \ldots, n \right\}.$$

From a result published by Tong [14] we know that if f is Schur-concave function of X then $P\left\{X \in A(a)\right\}$ is also a Schur-concave function of a.

Which means that if

$$a < b$$

then

$$P\{X \in A(a)\} \geq P\{X \in A(b)\}$$

But if we consider the volumes of the sets; volume of $A(a)$ would be larger than that of $A(b)$.

Hence the corresponding probability content could be larger. To overcome this situation it is suggested the inequalities via majorization

$$(\log a_1, \ldots, \log a_n) > (\log b_1, \ldots, \log b_n).$$

Such a majorization depends on the diversity of elements of a; where $\prod_{i=1}^{n} a_i$ is kept fixed. Hence the volumes of all sets under consideration of all inequalities would be equal.

The same concept applies to the n-dimensional ellipsoid

$$B(a) = \{x : \sum \left[\frac{x_i}{a_i} \right]^2 \leq 1 \}.$$

5.B THEORETICAL ASPECTS :

5.B.1. Arrangement Increasing Function :

5.B.1.1 Definition : Let $a = (a_1, \ldots, a_n)$ where $a_1 \leq a_2 \leq \ldots \leq a_n$.

We say that the function $f(a;x)$ is arrangement increasing if

(a) $f(a\pi; x\pi) = f(a; x)$ for all permutation matrices π and vectors x and a as given above and

(b) $f(a; x) \geq f(a; y)$ whenever $x >^t y$

where $x >^t y$ implies whenever x and y agree in all but two coordinates say i and j such that $i < j$

$x_i < x_j$ and $y_i = x_j$, $y_j = x_i$.

5.B.1.2 Example :

$$f(a;x) = (a_1 x_1)^2 + (a_2 x_2)^2 + (a_3 x_3)^2$$

Let $a_1 = 1,\ a_2 = 2\ ,\ a_3 = 3$

$x = (2,\ 4,\ 6) \quad y = (2,\ 6,\ 4)$

Observe that $x >^t y$.

$$f(a;x) = (1 \times 2)^2 + (2 \times 4)^2 + (3 \times 6)^2$$
$$= 392$$

$$f(a;y) = (1 \times 2)^2 + (2 \times 6)^2 + (3 \times 4)^2$$
$$= 292$$

Thus $f(a;x) > f(a;y)$

where $x >^t y$.

Hence $f(a;x)$ is arrangement increasing.

5.B.1.3 Remarks :

i) Any natural domain $\mathfrak{F} \subset R^n \times R^n$ for an AI function has the property that $(x,y) \in \mathfrak{F}$ implies $(x\,\pi^{(1)},\ y\,\pi^{(2)}) \in \mathfrak{F}$ for all permutation matrices $\pi^{(1)}$ and $\pi^{(2)}$.

ii) If $g_1, \ldots, g_k$ are AI functions on a set $\mathfrak{F}$ satisfying (i) and if $h : R^k \longrightarrow R$ is increasing in each argument, then the composition $h(g_1, \ldots, g_k)$ is an AI function on $\mathfrak{F}$.

iii) If g is AI on $R^n \times R^n$ and if $\Phi : R \longrightarrow R$, $\Psi : R \longrightarrow R$ are monotone in the same direction then g^* defined by $g^*(x;y) = g(\Phi(x_1),\ldots,\Phi(x_n);\Psi(x_1),\ldots,\Psi(x_n))$ is AI on $\mathcal{F}^* = \{(u,v) : u = (\Phi(x_1),\ldots,\Phi(x_n)), v = (\Psi(x_1),\ldots,\Psi(x_n))\}$ for some $(x,y) \in \mathcal{F}$.

iv) If g has the form $g(u,v) = \Phi(u+v)$ $u,v \in R^n$ then g is AI on $R^n \times R^n$; if and only if Φ is Schur-convex on R^n.

<u>Proof</u> : It is sufficient to prove this for $n = 2$

$x < y$ on R^2 if and only if $x\uparrow$ and $y\uparrow$ have the form

$$(x_{(1)}, x_{(2)}) = (r_2 + s_1, r_1 + s_2)$$

$$(y_{(1)}, y_{(2)}) = (r_1 + s_1, r_2 + s_2)$$

where $r_1 < r_2$, $s_1 < s_2$.

If g is AI on $R^n \times R^n$ it follows that

i) $g(r_1, r_2, s_2, s_1) = g(r_2, r_1; s_1, s_2)$

$\leq g(r_1, r_2; s_1, s_2) = g(r_2, r_1; s_2, s_1)$

ii) $\varphi(r_2 + s_2, r_2 + s_1) = \varphi(r_2 + s_1, r_1 + s_2)$

$\leq \varphi(r_1 + s_1, r_2 + s_2) = \varphi(r_2 + s_2, r_1 + s_1)$

Consequently φ is Schur-convex on R^2 conversily if φ is Schur-convex on R^2, then (ii) holds whenever

$r_1 < r_2$, $s_1 < s_2$ ie., (i) holds so g is AI on R^2.

iii) If g has the form

$g(u, v) = \varphi(u - v)$ for all $u, v \in R^n$, then g is AI on R^{2n}, if φ is Schur-concave on R^n.

Shaked and Tong's Theorem :

Let $(X_1, \ldots, X_n)$ have a density f and let A be a subset of R^n. If f and I_A (Indicator function of A) are such that $f(x/a_1, \ldots, x_n/a_n)$ and $I_A(x_1/a_1, \ldots, x_n/a_n)$ are AI in $a \in (0\ \infty)^n$ and $x \in R^n$, then $P(X_1/a_1, \ldots X_n/a_n)$ is Schur-concave in $(\log a_1, \ldots, \log a_n)$.

Outline of the proof :

Read $f(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n})$ as $g_1(a_1, \ldots, a_n;\ x_1, \ldots, x_n)$

and

$I_A(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n})$ as $g_2(x_1, \ldots, x_n;\ a_1, \ldots, a_n)$

Now write

$$g(a : b) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g_1(a;\ x)\, g_2(x;\ b)\, dx$$

Which is an AI function.

Through a transformation it is shown that g is of the form

$$g(a;\ b) = \left(\prod_{i=1}^{n} b_i\right) h\left(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n}\right) \qquad (1)$$

Now write

$$\tilde{h}(a;\ b) = h(\frac{b_1}{a_1}\ ,\ldots,\ \frac{b_n}{a_n})$$

Which is an AI function.

From this it follows that

$h(e^{b_1} / e^{a_1},\ldots, e^{b_n} / e^{a_n})$ is an AI function on R^n.

Hence $h(e^{c_1}\ ,\ldots,\ e^{c_n})$ is Schur-concave in $c \in R^n$.

i.e. the function $h(c_1,\ldots,c_n)$ is Schur-concave in $(\log c_1,\ldots,\ \log c_n)$.

Denote

$$\chi(a) = \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f(\frac{x_1}{a_1},\ldots,\ \frac{x_n}{a_n})\ \ I_A(x_1,\ldots,\ x_n)\ \ dx$$

Put $b_1 = \ldots = b_n = 1$ in (1) to obtain

$$\chi(a) = h(a_1^{-1},\ldots,\ a_n^{-1}).$$

Since $h(a)$ is Schur-Concave in $(\log a_1,\ldots,\ \log a_n)$, $\chi(a)$ is also Schur-Concave in $(\log a_1,\ldots,\ \log a_n)$.

i.e. $P\{\ (x_1/a_1,\ldots,\ x_n/a_n) \in A\ \}$ is Schur-concave in $(\log a_1,\ldots,\ \log a_n)$.

<u>Proof</u> : It is given that $f(x_1/a_1,\ldots,\ x_n/a_n)$ and $I_A(x_1/a_1,\ldots,\ x_n/a_n)$ are AI in $a \in (0\ \infty)^n$ and $x \in R^n$.

It is required to show that $P\{\ (x_1/a_1,\ldots,\ x_n/a_n) \in A\ \}$ is Schur-concave in $(\log a_1,\ldots,\ \log a_n)$.

i.e. $\int_A f(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n})\, dx$ is Schur-concave in $(\log a_1, \ldots, \log a_n)$.

i.e. $\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n})\, I_A(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n})\, dx$ is Schur-concave in $(\log a_1, \ldots, \log a_n)$.

Let φ and ψ be two n-variate real functions such that $g_1(a_1, \ldots, a_n;\, x_1, \ldots, x_n) \equiv \varphi(x_1/a_1, \ldots, x_n/a_n)$ is AI on $(0, \infty)^n \times R^n$.

and

$g_2(x_1, \ldots, x_n;\, a_1, \ldots, a_n) \equiv \psi(x_1/a_1, \ldots, x_n/a_n)$ is AI on $R^n \times (0, \infty)^n$.

Then

$$g(a : b) \equiv \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g_1(a;\, x)\, g_2(x;\, b)\, dx$$

is AI on $(0, \infty)^n \times (0, \infty)^n$.

(First note that

$$g(a\pi;\, b\pi) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g_1(a\pi;\, x)\, g_2(x;\, b\pi)\, dx$$

Rearranging the integrals to get

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g_1(a\pi;\, x\pi)\, g_2(x\pi;\, b\pi)\, dx\pi$$

$$= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g_1(a; x)\, g_2(x; b)\, dx$$

$$= g(a; b).$$

This satisfies condition (1) of an AI function.

Let π° be the permutation for which

$b\pi^\circ = (b_2, b_1, b_3, \dots, b_n)$ for all b. It is required to show that $g(a; b) \geq g(a; b\pi^\circ)$ when both vector a and b are arranged in increasing order.

Consider

$$g(a; b) - g(a; b\pi^\circ)$$

$$= \int \left[g_1(a; x)\, g_2(x; b) - g_1(a; x)\, g_2(x; b\pi^\circ)\right] dx$$

Break the region of integration into $x_1 < x_2$ and $x_1 > x_2$ and make a change in the variables of second region to obtain

$$g(a; b) - g(a, b\pi^\circ)$$

$$= \int_{x_1 < x_2} \Big[g_1(a; x)\, g_2(x; b) - g_1(a; x)\, g_2(x; b\pi^\circ)$$
$$+ g_1(a; x\pi^\circ)\, g_2(x\pi^\circ; b) - g_1(a; x\pi^\circ)\, g_2(x\pi^\circ; b\pi^\circ)\Big] dx$$

$$= \int_{x_1 < x_2} \Big[g_1(a; x)\, g_2(x; b) - g_1(a; x)\, g_2(x; b\pi^\circ)$$
$$+ g_1(a; x\pi^\circ)\, g_2(x\pi^\circ; b) - g_1(a; x\pi^\circ)\, g_2(x; b)\Big]$$

$$= \int_{x_1 < x_2} \left[g_1(a;\ x) - g_1(a;\ x\pi^0)\right] \left[g_2(x;\ b) - g_2(x;\ b\pi^0)\right] \ dx$$

as g_1 and g_2 are AI functions the integral is positive.

Hence $g(a;\ b)$ is AI.)

Substitute $y_i = x_i / b_i$

i.e., $b_i\ y_i = x_i$

i.e., $b_i\ dy_i = dx_i$

Hence $\prod_{i=1}^{n} dx_i = \prod_{i=1}^{n} b_i \prod_{i=1}^{n} dy_i$

Also on integration the result depends only on $\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n}$.

Thus $g(a;\ b)$ is written as a product of $\Pi\ b_i$ and

$h(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n})$ for some function h on $(0\ \infty)^n$.

$$g(a;\ b) = \prod_{i=1}^{n} b_i \ \ h(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n}) \tag{2}$$

The function $\tilde{h}$ defined by

$\tilde{h}(a;\ b) = h(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n})$ is AI on $(0\ \infty)^n \times (0\ \infty)^n$.

To verify this write

$$\tilde{h}(a;\ b) = (\prod_{i=1}^{n} b_i)^{-1}\ g(a;\ b)$$

$$\tilde{h}(a\pi,\ b\pi) = (\prod_{i=1}^{n} - b_i)^{-1}\ g(a\pi;\ b\pi)$$

$$= \left(\prod_{i=1}^{n} b_i\right)^{-1} g(a;\ b) \qquad \text{(since } g \text{ is AI)}$$

$$= \tilde{h}(a;\ b).$$

Let $\quad b >^t c$

$$\tilde{h}(a;\ c) = \left(\prod_{i=1}^{n} c_i\right)^{-1} g(a;\ c)$$

$$= \left(\prod_{i=1}^{n} b_i\right)^{-1} g(a;\ c) \quad \left(\text{since } \prod_{i=1}^{n} c_i = \prod_{i=1}^{n} b_i\right)$$

$$< \left(\prod_{i=1}^{n} b_i\right)^{-1} g(a;\ b) \qquad \text{(since } g \text{ is AI)}$$

$$= \tilde{h}(a;\ b).$$

Thus $\tilde{h}(a;\ b)$ is AI on $(0,\ \infty)^n \times (0,\ \infty)^n$

since $h(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n})$ is AI on $(0,\ \infty)^n \times (0,\ \infty)^n$ it follows that

$$h(e^{b_1} / e^{a_1}, \ldots, e^{b_1} / e^{a_n}) \text{ is AI on } R^n$$

(follows from 5.B.1.3 (i))

Hence $h(e^{c_1}, \ldots, e^{c_n})$ is Schur-concave in $c \in R^n$

(follows from 5.B.1.3 (iv))

i.e., $h(c_1, \ldots, c_n)$ is Schur-concave in $(\log c_1, \ldots, \log c_n)$

Denote

$$\chi(a) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi\left(\frac{x_1}{a_1}, \ldots, \frac{x_n}{a_n}\right) \psi(x_1, \ldots, x_n)\, dx.$$

Put $b_1 = b_2 = \ldots = b_n = 1$ in (2) to obtain

$$\chi(a) = h(a_1^{-1}, \ldots, a_n^{-1})$$

Since $h(a)$ is Schur-concave in $(\log a_1, \ldots, \log a_n)$ it follows that $\chi(a)$ is Schur-concave in $(\log a_1, \ldots, \log a_n)$ also.

By choosing $\varphi(x) = f(x)$ and

$\psi(x) = I_A(x)$ we would arrive at the argument aspaired in (1).

Hence the theorem.

<u>5.B.3 Remark</u> : Similar results due to Anderson; Marshall and Olkin mainly depends on conditions like unimodality and Schur-concavity. This is evident from the discussions in the previous chapters. Now it is natural to seek similar set up for Shaked and Tong's theorem. Moreover it is more easy to check unimodality and Schur-concavity rather than AI property. For this very prupose theorem 5.B.5 is presented.

<u>5.B.4 Definition</u> : A random vector X (or its distribution) is called monotone unimodel if for every convex set $c \subset R^n$ and every $x \neq 0$, the quantity $P\{X \in c + k x\}$ is non-decreasing in $k \geq 0$.

<u>5.B.5 Theorem</u> : If (X_1, X_2) with a Schur-concave density $f(x_1, x_2)$ is monotone unimodel, if $f(x_1, -x_2)$ is Schur-

concave and if $A \subset R^2$ is measurable symmetric (about 0) permutation invariant and convex, then $P\{X_1/a_1, X_2/a_2) \in A\}$ is Schur-concave in $(\log a_1, \log a_2)$.

Outline of the proof :

It is known that every monotone unimodel random vector is symmetric about origin. Hence the monotone unimodel function of the theorem is symmetric about $\{(x_1, x_2) : x_1 = x_2\}$ and $\{(x_1, x_2) : x_1 + x_2 = 0\}$. This would mean that $f(x_1/a_1, x_2/a_2)$ cannot be AI in $a \in (0\ \infty)^2$ and $x \in R^2$. But f could be restricted to the region $\{(x_1, x_2) : x_1 + x_2 \geq 0\}$ which is equivalent to the conditional density of (X_1, X_2) given that $X_1 + X_2 \geq 0$ being AI. By unristricting we would arrive at the result.

Proof :

Define

$$\tilde{A} = A \cap \{(x_1, x_2) : x_1 + x_2 \geq 0\}$$

Initially we would show that

$$P\{(\frac{1}{a_1} X_1, \frac{1}{a_2} X_2) \in A / X_1 + X_2 \geq 0\} \text{ is Schur-}$$

concave in $(\log a_1, \log a_2)$ (1)

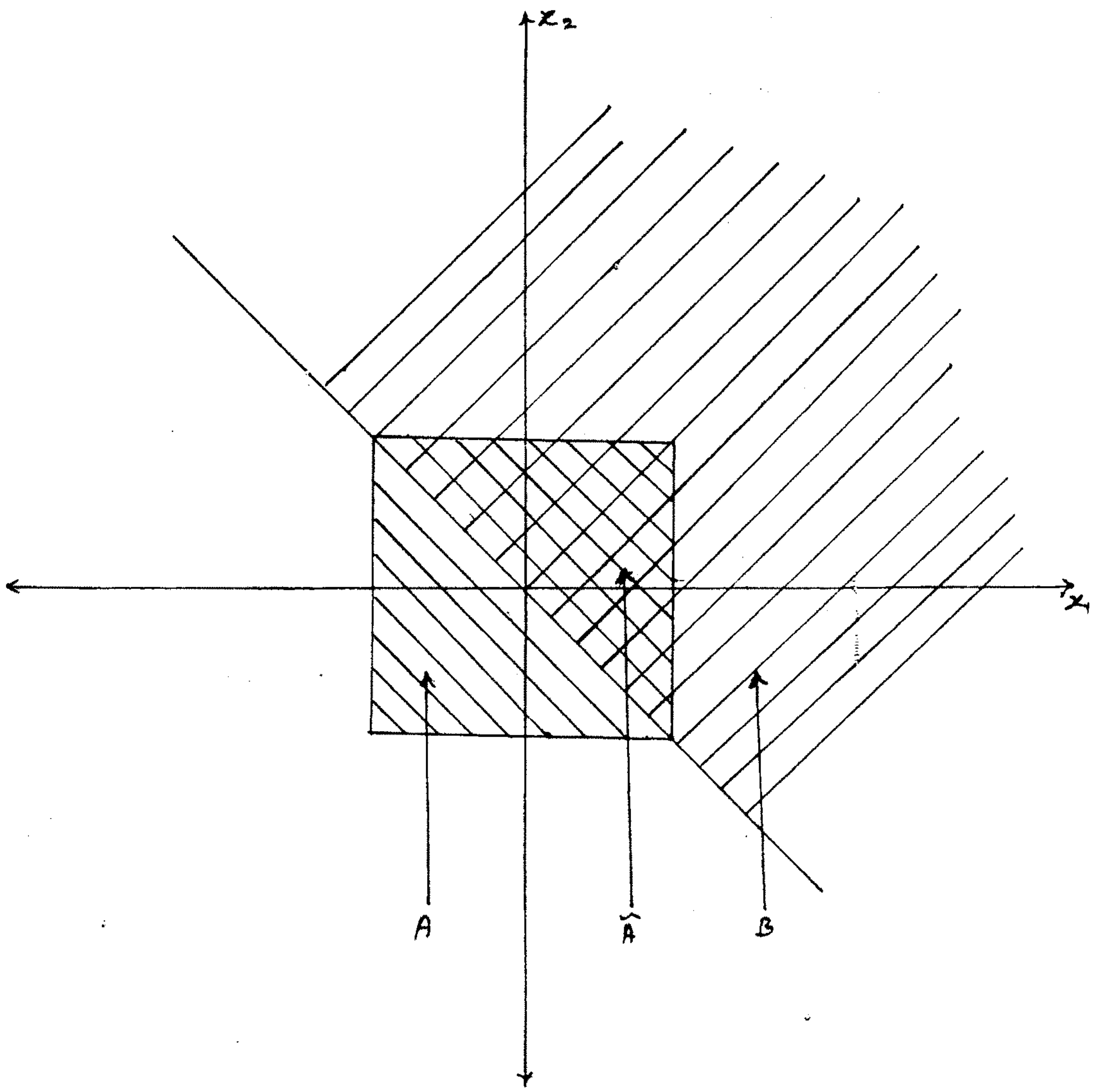

fig. 1

As $f(x_1, x_2)$ is symmetric about

$\{(x_1, x_2) : x_1 + x_2 = 0\}$ it could be shown that

$P\{(\frac{1}{a_1} X_1, \frac{1}{a_2} X_2) \in A\}$ is Schur- concave in $(\log a_1, \log a_2)$

Define $B = \{(x_1, x_2) : x_1 + x_2 \geq 0\}$

To prove (1) it suffices to show that

$g(x_1, x_2) \equiv 2\, f(x_1, x_2)\, I_B(x_1, x_2)$ and $I_{\tilde{A}}$ satisfy

$g(\frac{x_1}{a_1}, \ldots, \frac{x_2}{a_2})$ is AI in $a \in (0, \infty)^2$ and $x \in R^2$ and

$I_{\tilde{A}} (\frac{x_1}{a_1}, \ldots, \frac{x_2}{a_2})$ is AI in $a \in (0, \infty)^2$ and $x \in R^2$.

To show that g is AI it is sufficient to show that f is AI.

i.e., $f(\frac{x_1}{a_1}, \ldots, \frac{x_2}{a_2}) \leq f(\frac{x_1}{a_1}, \ldots, \frac{x_2}{a_2})$

whenever $0 < a_1 < a_2$ and $x_1 + x_2 \geq 0$

put $\frac{1}{a_1} = c_1$ and $\frac{1}{a_2} = c_2$

i.e., $f(c_1 x_1, c_2 x_2) \leq f(c_2 x_1, c_1 x_2)$

whenever $c_1 > c_2 > 0$ and $x_1 + x_2 \geq 0$

<u>Case - 1</u> :

Let $x_1 \geq x_2 > 0$

Denote

$(y_1, y_2) = (c_1 x_1 + c_2 x_2) / (c_2 x_1 + c_1 x_2) (c_2 x_1, c_1 x_2)$

It is easy to varify that

$$(c_1 x_1, c_2 x_2) > (y_1, y_2)$$

Hence

$f(c_1 x_1, c_2 x_2) \leq f(y_1 y_2)$ (By Schur-concavity)

$\leq f(c_2 x_1, c_1 x_2)$ (By monotone unimodality)

Hence the result.

<u>Case - 2</u> :

$$x_1 \geq 0 > x_2 \quad (\text{and} \quad x_1 + x_2 \geq 0)$$

Since (X_1, X_2) is monotone unimodal it follows that $(X_1, -X_2)$ is also monotone unimodal. It's density h is given by

$$h(x_1, x_2) = f(x_1, -x_2)$$

By assumption the density of $(X_1, -X_2)$ is Schur-concave.

Hence it follows from the preceeding argument that when $x_1 \geq -x_2 > 0$

$$h(c_1 x_1, -c_2 x_2) \leq h(c_2 x_1, -c_1 x_2)$$

i.e., $\quad f(c_1 x_1, c_2 x_2) \leq f(c_2 x_1, c_1 x_2)$

as was to be shown.

<u>Case - 3</u> :

$x_2 \geq x_1 > 0$ (and $x_1 + x_2 \geq 0$) it can be proved on similar lines as above.

Hence f is AI function.

Hence $g(\frac{x_1}{a_1}, \frac{x_2}{a_2})$ is AI in $a \in (0\ \infty)^2$ and $x \in R^2$.

Similarly,

$$I_{\tilde{A}}(\frac{x_1}{a_1}, \frac{x_2}{a_2}) \text{ is AI in } a \in (0\ \infty)^2 \text{ and } x \in R^2.$$

Hence the theorem.

<u>5.B.6 Remark</u> : It could be noted that the class of density functions in theorem 5.B.6 is a subclass of Schur-convex functions. The additional condition is symmetry (about 0) and unimodality. If either of these conditions fail; we fail to yield the result. This is illustrated in an example below.

5.B.7 Example :

Let $X = (X_1, X_2)$ have a uniform density over the region.

$$\{(x_1, x_2) : |x_1 - x_2| \le 6, \quad 2 \le |x_1 + x_2| \le 6\}$$

which is a Schur-concave function of x.
Then the probability content of A(a) is zero for $a = (1, 1)$ and is positive for all a satisfying

$$a_1 = a_2^{-1} \ne 1.$$

From fig.2 one can observe that the assumption of unimodality does not satisfy this density function.

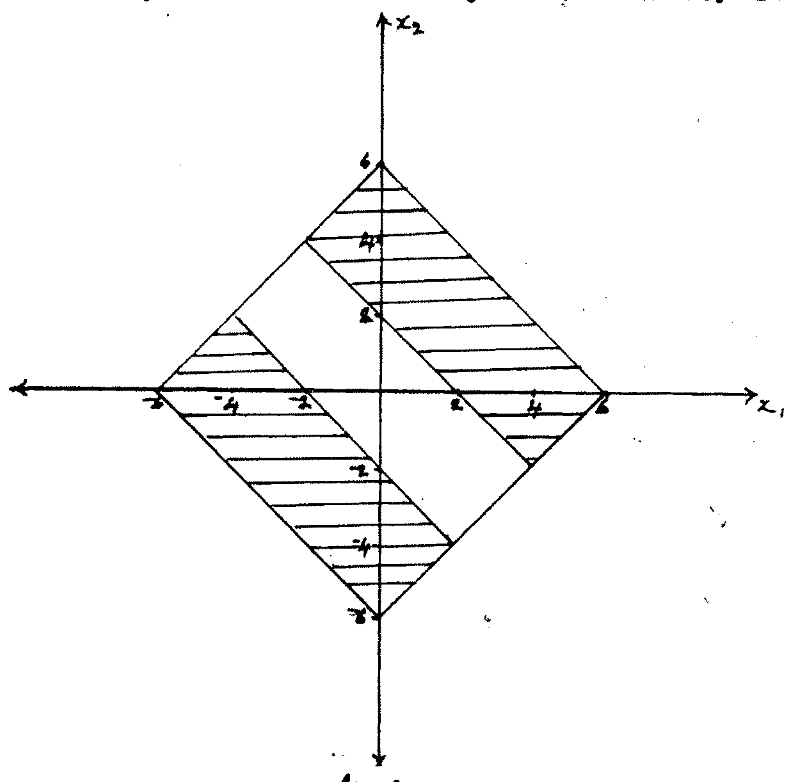

fig.2

Hence it failed to yield the aspaired result.