CHAPTER - IV

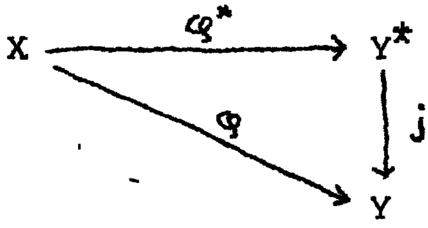
CLEAVAGES AND OPCLEAVAGES IN THE CATEGORY OF ANTI-COMMUTATIVE GRADED ALGEBRAS.

In the last two chapters we have discussed cleavages and opcleavages in the category of derivation modules and the category of complexes. In this chapter we shall develop analogous concepts for the anticommutative graded algebras. Since the proofs of the propositions and theorems are very similar to those of the corresponding Propositions and Theorems of the previous chapter, we have omitted the proofs.

Definition (4.1) : Let $X = \sum_{n \geq 0} X_n$ be an anticommutative graded R-algebra. If $X_0 = A$ then we call X an anticommutative graded A-algebra. Let X be anticommutative graded A-algebra and Y the anticommutative graded B-algebra and $f : A \longrightarrow B$ be an algebra homomorphism in $\mathcal{A}$, then a graded algebra homomorphism $\varphi : X \longrightarrow Y$ is said to be f-graded algebra homomorphism if and only if $\varphi | A = f$. In this case Y is said to be φ - simple if and only if $B \cup \varphi(X)$ generates Y as an R-algebra.

Proposition (4.1) : Let X be an anticommutative graded A-algebra. Then for any anticommutative graded B-algebra Y and f-graded algebra homomorphism $\varphi : X \longrightarrow Y$, there exists a φ^*-simple anticommutative graded B-algebra Y^* and a graded B-algebra monomorphism $j : Y^* \longrightarrow Y$ such that $j \varphi^* = \varphi$.

Here Y^* is generated by B U φ (X), $\varphi^* = \varphi$, then clearly Y^* is φ^*-isimple and j : $Y^* \longrightarrow Y$ is inclusion mapping making the following diagram commutative.



Proposition (4.2) : Let f: A$\longrightarrow$B be a morphism in $\mathcal{A}$. For any anticommutative graded A-algebra X there exists an anticommutative graded B-algebra X' and f-graded algebra homomorphism $(\psi_f)_X$: X$\longrightarrow$X' in $\mathcal{G}$ such that for any anticommutative graded B-algebra Y and any f-graded algebra homomorphism φ : X$\longrightarrow$Y, there exists a unique graded B-algebra homomorphism φ'' : X'$\longrightarrow$Y satisfying $\varphi'' (\psi_f)_X = \varphi$. Moreover X' and $(\psi_f)_X$ are unique in the sense that if there exists another such anticommutative graded B-algebra $\bar{X}$ and f-graded algebra homomorphism h_X : X$\longrightarrow\bar{X}$, then there exists a graded B-algebra isomorphism i : X'$\longrightarrow\bar{X}$ satisfying i $(\psi_f)_X = h_X$.

Proof is very similar to that of Prop (3.2). Here by a graded B-algebra homomorphism φ : X$\longrightarrow$Y between the anticommutative graded B-algebras, we mean a graded algebra homomorphism which maps B identically.

Proposition (4.3) : Let X and Y be anticommutative graded A-algebras and X' and Y' be the corresponding anticommutative graded B-algebras. Then for any graded A-algebra homomorphism $\lambda : X \longrightarrow Y$ there exists a unique graded B-algebra homomorphism $\lambda' : X' \longrightarrow Y'$ such that $\lambda' (\psi_f)_X = (\psi_f)_Y \lambda$.

The proof is very similar to that of Prop (3.3).

Let $\mathcal{G}(A)$ denote the category of all anticommutative graded A-algebras and graded A-algebra homomorphisms, where A is any unitary commutative algebra. Let A and B be unitary commutative algebras and $f: A \longrightarrow B$ an algebra homomorphism. Define $f_* : \mathcal{G}(A) \longrightarrow \mathcal{G}(B)$ as $f_*(X) = X'$ [as defined in Prop (4.2)] and $f_*(\lambda) = \lambda'$ [as defined in Prop. (4.3)]. Then the following Theorem can similarly be proved.

Theorem (4.1) : If $f: A \longrightarrow B$ is a morphism in $\mathcal{A}$, then there exists a covariant functor $f_*: \mathcal{G}(A) \longrightarrow \mathcal{G}(B)$ defined as $f_*(X) = X'$ and $f_*(\lambda) = \lambda'$.

Proposition (4.4) : Let A, B, C be unitary commutative R-algebras and $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be algebra homomorphisms. Then there exists a natural equivalence $C_{fg} : (g\, f)_* \longrightarrow g_*\, f_*$.

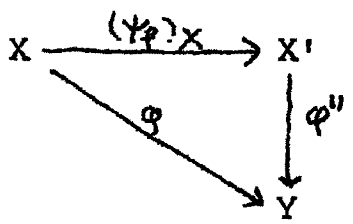
Proof is similar to that of Prop (3.4).

Let $\mathcal{A}$ denote the category of unitary commutative R-algebras and let $\mathcal{G}$ denote the category of all anticommutative graded algebras over R and graded algebra homomorphisms. Consider the functor

$P : \mathcal{G} \longrightarrow \mathcal{A}$ defined by $P(X) = X_o$ and $P(\varphi) = \varphi_o$. Then the fibre $P^{-1}(A)$ is the category $\mathcal{G}(A)$ of all anticommutative graded A-algebras and graded A-algebra homomorphisms. Let $J_A : \mathcal{G}(A) \to \mathcal{G}$ denote the inclusion functor. Now our claim is :

Theorem (4.2) : The functor $P : \mathcal{G} \to \mathcal{A}$ admits an opcleavage $\{f_*, \psi_f, c_{fg}\}$.

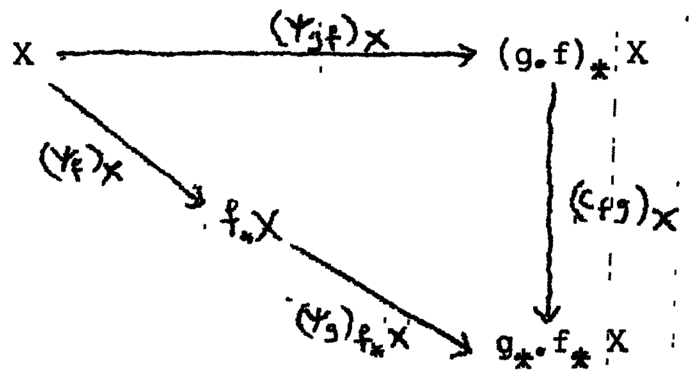
Proof : For each morphism $f : A \longrightarrow B$ in $\mathcal{A}$ and any X in $\mathcal{G}(A)$ there exists a unique X' in $\mathcal{G}(B)$ and f - graded algebra homomorphism $(\psi_f)_X : X \longrightarrow X'$ in $\mathcal{G}$ by Prop (4.2). For any morphism $\lambda : X \longrightarrow Y$ in $\mathcal{G}(A)$, there exists a unique morphism $\lambda' : X' \longrightarrow Y'$ in $\mathcal{G}(B)$ by Prop (4.3). Thus each $f: A \longrightarrow B$ in $\mathcal{A}$ gives rise to a functor $f_*: \mathcal{G}(A) \longrightarrow \mathcal{G}(B)$. There exists a natural transformation $\psi_f : J_A \longrightarrow J_B f_*$ satisfying the condition that $P((\psi_f)_X) = f$ by Props (4.2), (4.3). For any f-graded algebra homomorphism $\varphi : X \longrightarrow Y$ satisfying $P(\varphi) = f$, there exists a unique graded B-algebra homomorphism $\varphi'' : X' \longrightarrow Y$ such that $\varphi'' (\psi_f)_X = \varphi$ by Prop (4.2), i.e. making the following diagram commutative.



Consider the composition $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. Then for each $X \in \mathcal{G}(A)$ there exists by Prop (4.4) a uniquely determined morphism

$$(c_{fg})_X : (g\ f)_* X \longrightarrow g_*\ f_*\ X \quad \text{in} \quad \mathcal{G}(C)$$

such that $(c_{fg})_X\ (\psi_{gf})_X = (\psi_g)_{f_* X}\ (\psi_f)_X$ making the following diagram commutative.



It is easily seen that $(c_{fg})_X$ are the components of a natural transformation

$$c_{fg} : (g\ f)_* \longrightarrow g_*\ f_*$$

and it follows from Prop (4.4) that each such c_{fg} is a natural equivalence. Thus the functor $P: \mathcal{G} \to \mathcal{A}$ admits an opcleavage $\{f_*, \psi_f, c_{fg}\}$.

In the following we shall prove that each morphism $f: A \longrightarrow B$ in $\mathcal{A}$ gives rise to a covariant functor $f^*: \mathcal{G}(B) \longrightarrow \mathcal{G}(A)$.

<u>Proposition (4.5)</u> : Let $f : A \longrightarrow B$ be a morphism in $\mathcal{A}$. Then for any anticommutative graded B-algebra Y, there exists a unique anticommutative graded A-algebra $\overline{Y}$ and the f-graded algebra homomorphism $(\theta_f)_Y : \overline{Y} \longrightarrow Y$.

Proof is analogous to that of Prop (3.5).

<u>Proposition (4.6)</u> : Let X be an anticommutative graded A-algebra and Y an anticommutative graded B-algebra. Then for any f-graded algebra homomorphism $\varphi : X \longrightarrow Y$, there exists a unique graded A-algebra homomorphism $\varphi' : X \longrightarrow \overline{Y}$ such that $(\theta_f)_Y \quad \varphi' = \varphi$.

Proof is analogous to that of Prop (3.6).

<u>Proposition (4.7)</u> : Let X and Y be anticommutative graded B-algebras and $\overline{X}$ and $\overline{Y}$ be the corresponding anticommutative graded A-algebras. Then for any morphism $k : X \longrightarrow y$ in $\mathcal{G}(B)$, there exists a unique morphism $\overline{k} : \overline{X} \longrightarrow \overline{Y}$ in $\mathcal{G}(A)$ such that $(\theta_f)_Y \overline{k} = k \ (\theta_f)_X$.

Define the mapping $f^*: \mathcal{G}(B) \longrightarrow \mathcal{G}(A)$ as $f^*(X) = \overline{X}$ and $f^*(k) = \overline{k}$ for all $X \in \mathcal{G}(B)$ and $k \in \mathcal{G}(B)$. Then the followina Theorem can similarly be proved as Theorem (3.3).

<u>Theorem (4.3)</u> : If $f:A \longrightarrow B$ is a morphism in $\mathcal{A}$, then there exists a covariant functor $f^* : \mathcal{G}(B) \longrightarrow \mathcal{G}(A)$.

Proposition (4.8) : Let A, B, C be unitary commutative R-algebras and $f:A \longrightarrow B$ and $g:B \longrightarrow C$ be algebra homomorphisms. Then $f^* g^* = (g.f)^*$

Proof is analogous to that of Prop (3.8).

Theorem (4.4) : The functor $P : \mathcal{G} \longrightarrow \mathcal{A}$ admits a normalized split cleavage $\{f^*, \theta_f, d_{fg}\}$.

Proof is very similar to that of Theorem (3.4).