CHAPTER - V

Let $\mathcal{C}$ denote the category of all R-complexes and $\mathcal{G}$ denote the category of all anticommutative graded algebras. Consider the forgetful functor

$$U : \mathcal{C} \longrightarrow \mathcal{G}$$

defined by $U(X,d) = X$ for all $(X,d) \in \mathcal{C}$

We claim :

<u>Theorem (5.1)</u> : The functor $U : \mathcal{C} \to \mathcal{G}$ is cleavage preserving.

<u>Proof</u> : Consider a commutative diagram of functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "U"] \arrow[dr, "P"'] & & \mathcal{G} \arrow[dl, "\widetilde{P}"] \\ & \mathcal{A} & \end{tikzcd}$$

Where the functors P and $\widetilde{P}$ have cleavages $\{f^*, \theta_f, d_{fg}\}$ and $\{\widetilde{f}^*, \widetilde{\theta}_f, \widetilde{d}_{fg}\}$ respectively. Let $U | \mathcal{C}(A) = U_A$ for all $A \in \mathcal{A}$.

Let $(X,d) \in \mathcal{C}(B)$ be any object. Then $U_A f^*(X,d) = U_A(\overline{X},\overline{d}) = \overline{X} = A \oplus \sum_{n \geq 1} X_n$ by Prop (3.5). $\widetilde{f}^* U_B(X,d) = \widetilde{f}^* X = \overline{X} = A \oplus \sum_{n \geq 1} X_n$ by Prop (4.5). Hence the following diagram commutes.

$$\begin{tikzcd} \mathcal{C}(B) \arrow[r, "U_B"] \arrow[d, "f^*"'] & \mathcal{G}(B) \arrow[d, "\widetilde{f}^*"] \\ \mathcal{C}(A) \arrow[r, "U_A"] & \mathcal{G}(A) \end{tikzcd}$$

It follows obviously that $U_A f^* = \tilde{f}^* U_B$.

For every $(X,d) \in \mathcal{C}(B)$, $f:A \longrightarrow B$ induces

$$U((\theta_f)_X) : U_A f^*(X,d) \longrightarrow U_B(X,d) \text{ in } \mathcal{G}$$

By Axiom (1) of cleagave, there exists a unique morphism

$$(\eta_f)_X : U_A f^*(X,d) \longrightarrow \tilde{f}^* U_B(X,d)$$

making the following diagram commutative :

$$\begin{tikzcd}
U_A f^*(X,d) \arrow[d, "(\eta_f)_X"'] \arrow[dr, "U((\theta_f)_X)"] & \\
f^* U_B(X,d) \arrow[r, "(\tilde{\theta}_f)_{UX}"'] & U_B(X,d)
\end{tikzcd}$$

It is obvious that $(\eta_f)_X$ is the identity on $U_A f^*(X,d) = \tilde{f}^* U_B(X\ d)$.

Therefore, we have $U(\theta_f) = \tilde{\theta}_f$.

It also follows obviously that $U(d_{fg}) = \tilde{d}_{fg}$. Hence the functor U is cleavage - preserving.

<u>Theorem (5.2)</u> : The functor $U : \mathcal{C} \to \mathcal{G}$ is opcleavage-preserving.

<u>Proof</u> : Consider a commutative diagram of functors -

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "U"] \arrow[dr, "P"'] & & \mathcal{G} \arrow[dl, "\tilde{P}"] \\
& \mathcal{A} &
\end{tikzcd}$$

Where the functors P and $\tilde{P}$ have opcleavages $\{f_*, \psi_f, c_{fg}\}$ and $\{\tilde{f}_*, \tilde{\psi}_f, \tilde{c}_{fg}\}$ respectively.

Let $U \mid \mathcal{C}(A) = U_A$ for all $A \in \mathcal{A}$.

Let $(X, d) \in \mathcal{C}(A)$ be any object. Then

$$\tilde{f}_* U_A(X,d) = \tilde{f}_* X = X' \qquad \text{by Prop (4.2).}$$

$$U_B f_* (X,d) = U_B (X',d') = X' \qquad \text{by Prop (3.2).}$$

Hence, we have the following diagram commutative

$$\begin{tikzcd} \mathcal{C}(A) \arrow[r, "U_A"] \arrow[d, "f_*"'] & \mathcal{G}(A) \arrow[d, "\tilde{f}_*"] \\ \mathcal{C}(B) \arrow[r, "U_B"] & \mathcal{G}(B) \end{tikzcd}$$

It follows obviously that $\tilde{f}_* U_A = U_B f_*$.

For every $(X,d) \in \mathcal{C}(A)$, $f : A \longrightarrow B$ induces

$$U((\psi_f)_X) : U_A(X,d) \longrightarrow U_B f_*(X,d) \text{ in } \mathcal{G}.$$ By Axiom (1) of opcleavage, there exists a unique morphism

$$(\eta_f)_X : \tilde{f}_* U_A (X,d) \longrightarrow U_B f_* (X,d)$$

making the following diagram commutative.

$$\begin{tikzcd} U_A(X,d) \arrow[r, "(\tilde{\psi}_f)_{UX}"] \arrow[dr, "U((\psi_f)_X)"'] & \tilde{f}_* U_A(X,d) \arrow[d, "(\eta_f)_X"] \\ & U_B f_*(X,d) \end{tikzcd}$$

It is obvious that $(\eta_f)_X$ is the identity on

$$\tilde{f}_* U_A (X,d) = U_B f_* (X,d).$$

Therefore, we have $U(\psi_f) = \tilde{\psi}_f$.

It also follows obviously that $U(C_{fg}) = \tilde{C}_{fg}$.

Hence, the functor U is opcleavage-preserving.

Let $\mathcal{D}$ denote the category of all R-derivation modules. Consider the projection functor

$$T : \mathcal{C} \longrightarrow \mathcal{D}$$

defined by $T(X,d) = (X_o, X_1, d_o)$ for all $(X,d) \in \mathcal{C}$.

We claim :

<u>Theorem (5.3)</u> : The functor $T : \mathcal{C} \to \mathcal{D}$ is cleavage-preserving.

<u>Proof</u> : Consider a commutative diagram of functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "T"] \arrow[dr, "P"'] & & \mathcal{D} \arrow[dl, "\tilde{P}"] \\ & \mathcal{A} & \end{tikzcd}$$

Where the functors P and $\tilde{P}$ have cleavages $\{f^*, \theta_f, d_{fg}\}$ and $\{\tilde{f}^*, \tilde{\theta}_f, \tilde{d}_{fg}\}$ respectively.

Let $T \mid \mathcal{C}(A) = T_A$ for all $A \in \mathcal{A}$.

Let $(X,d) \in \mathcal{C}(B)$ be any object. Then

$T_A f^*(X,d) = T_A(\bar{X},\bar{d}) = (\bar{X}_o, \bar{X}_1, \bar{d}_o) = (A, \bar{X}_1, \bar{d}_o)$ by Prop (3.5)

$\tilde{f}^* T_B(X,d) = \tilde{f}^*(X_o,X_1,d_o) = (A,\bar{X}_1, \bar{d}_o)$ by Prop (2.5).

Hence the following diagram commutes.

$$\begin{tikzcd} \mathcal{C}(B) \arrow[r, "T_B"] \arrow[d, "f^*"'] & \mathcal{D}(B) \arrow[d, "\tilde{f}^*"] \\ \mathcal{C}(A) \arrow[r, "T_A"] & \mathcal{D}(A) \end{tikzcd}$$

It follows obviously that $T_A\ f^* = \tilde{f}^*\ T_B$.

For every $(X,d) \in \mathcal{C}(B)$, $f: A \longrightarrow B$ induces

$$T\ ((\theta_f)_X)\ :\ T_A\ f^*(X,d) \longrightarrow T_B(X,d)\ .$$

in $\mathcal{D}$. By Axiom (i) of cleavage, there exists a unique morphism

$$(\eta_f)_X\ :\ T_A\ f^*(X,d) \longrightarrow \tilde{f}^*\ T_B\ (X,d)$$

making the following diagram commutative :

$$\begin{tikzcd} T_A\ f^*\ (X,d) \arrow[d, "(\eta_f)_X"'] \arrow[drr, "T((\theta_f)_X)"] & & \\ f^*\ T_B(X,d) \arrow[rr, "(\tilde{\theta}_f)_{TX}"'] & & T_B(X,d) \end{tikzcd}$$

It is obvious that $(\eta_f)_X$ is the identity on

$$T_A\ f^*(X,d) = \tilde{f}^*\ T_B(X,d).$$

Therefore, we have $T\ (\theta_f) = \tilde{\theta}_f$.

It also follows obviously that $T(d_{fg}) = \tilde{d}_{fg}$. Thus the functor T is cleavage-preserving.

Theorem (5.4) : The functor $T : \mathcal{C} \to \mathcal{D}$ is opcleavage-preserving.

Proof : Consider a commutative diagram of functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "T"] \arrow[dr, "P"'] & & \mathcal{D} \arrow[dl, "\widetilde{P}"] \\ & \mathcal{A} & \end{tikzcd}$$

Where the functors P and $\widetilde{P}$ have opcleavages $\{f_*, \psi_f, c_{fg}\}$ and $\{\widetilde{f}_*, \widetilde{\psi}_f, \widetilde{c}_{fg}\}$ respectively.

Let $(X,d) \in \mathcal{C}(A)$ be any object. Then $\widetilde{f}_* T_A(X,d) = \widetilde{f}_*(A,X_1,d_o) =$ $= (B, X_1', d_o')$ by Prop (2.2). $T_B f_* (X,d) = T_B(X', d') =$ $= (B, X_1', d_o')$ by Prop (3.2).

Hence, we have the following diagram commutative.

$$\begin{tikzcd} \mathcal{C}(A) \arrow[r, "T_A"] \arrow[d, "f_*"'] & \mathcal{D}(A) \arrow[d, "\widetilde{f}_*"] \\ \mathcal{C}(B) \arrow[r, "T_B"] & \mathcal{D}(B) \end{tikzcd}$$

It follows obviously that $\widetilde{f}_* T_A = T_B f_*$.

For every $(X,d) \in \mathcal{C}(A)$, $f : A \longrightarrow B$ induces

$$T((\psi_f)_X) : T_A(X,d) \longrightarrow T_B f_*(X,d)$$

in $\mathcal{D}$. By Axiom (I) of opcleavage, there exists a unique morphism

$$(\eta_f)_X : \widetilde{f}_* T_A (X,d) \longrightarrow T_B f_* (X,d)$$

making the following diagram commutative

$$\begin{tikzcd}
T_A(X,d) \arrow[r, "(\widetilde{\psi}_f)_{TX}"] \arrow[dr, "T((\psi_f)_X)"'] & \widetilde{f}_* T_A(X,d) \arrow[d, "(\eta_f)_X"] \\
 & T_B f_*(X,d)
\end{tikzcd}$$

It is obvious that $(\eta_f)_X$ is the identity on $\widetilde{f}_* T_A(X,d) = T_B f_*(X,d)$.

Therefore, we have $T(\psi_f) = \widetilde{\psi}_f$.

It also follows obviously that $T(c_{fg}) = \widetilde{c}_{fg}$.

Hence, the functor T is opcleavage - preserving.

By Remark (9) of Chapter II, it follows that every R-derivation module (A, M, d) can be considered as an R-complex (X,δ) where $X_o = A$, $X_1 = M$, $X_n = 0$ for $n \geqslant 2$, and $\delta_o = d$, $\delta_n = 0$ for $n \geqslant 1$.

Consider the inclusion functor

$$I : \mathcal{D} \longrightarrow \mathcal{C}$$

defined as above. We claim :

<u>Theorem (5.5)</u> : The functor $I : \mathcal{D} \to \mathcal{C}$ is cleavage-preserving.

<u>Proof</u> : Consider a commutative diagram of functors -

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "I"] \arrow[dr, "P"'] & & \mathcal{C} \arrow[dl, "\widetilde{P}"] \\
 & \mathcal{A} &
\end{tikzcd}$$

Where the functors P and $\widetilde{P}$ have cleavages $\{f^*, \theta_f, d_{fg}\}$ and $\{\tilde{f}^*, \tilde{\theta}_f, \tilde{d}_{fg}\}$ respectively.

Let $I \mid \mathcal{D}(A) = I_A$ for all $A \in \mathcal{A}$

Let $(B, M, d) \in \mathcal{D}(B)$ be any object. Then

$I_A \quad f^*(B,M,d) = I_A(A,\overline{M},\overline{d}) = (A, \overline{M}, \overline{d})$ by Prop (2.5)

$\tilde{f}^* \quad I_B\,(B, M, d) = \tilde{f}^*\,(B, M, d) = (A, \overline{M}, \overline{d})$ by Pro. (3.5)

Hence the following diagram commutes.

$$\begin{tikzcd}
\mathcal{D}(B) \arrow[r, "I_B"] \arrow[d, "f^*"'] & \mathcal{C}(B) \arrow[d, "\tilde{f}^*"] \\
\mathcal{D}(A) \arrow[r, "I_A"] & \mathcal{C}(A)
\end{tikzcd}$$

It follows obviously that $I_A\, f^* = \tilde{f}^*\, I_B$.

For every $(B, M, d) \in \mathcal{D}(B)$, $f : A \longrightarrow B$ induces

$$I((\theta_f)_M) : I_A\, f^*(B, M, d) \longrightarrow I_B(B,M,d)$$

in $\mathcal{C}$. By Axiom (1) of cleavage, there exists a unique morphism,

$$(\eta_f)_M : I_A\, f^*\,(B,M,d) \longrightarrow \tilde{f}^*\, I_B\,(B,M,d)$$

making the following diagram commutative

$$\begin{tikzcd}
I_A\, f^*\,(B, M, d) \arrow[d, "(\eta_f)_M"'] \arrow[dr, "I((\theta_f)_M)"] & \\
\tilde{f}^*\, I_B(B,M,d) \arrow[r, "(\tilde{\theta}_f)_{IM}"] & I_B(B,M,d)
\end{tikzcd}$$

It is obvious that $(\eta_f)_M$ is the identity on $I_A . f^*(B,M,d) =$
$= \widetilde{f}^* . I_B(B, M, d)$.

Therefore, we have $I(\theta_f) = \widetilde{\theta}_f$.

It also follows obviously that $I(d_{fg}) = \widetilde{d}_{fg}$. Thus the functor I is cleavage-preserving.

<u>Thecrem (5.6)</u> : The functor $I : \mathcal{D} \to \mathcal{C}$ is opcleavage-preserving.

<u>Proof</u> : Consider a commutative diagram of functors

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "I"] \arrow[dr, "P"'] & & \mathcal{C} \arrow[dl, "\widetilde{P}"] \\
& \mathcal{A} &
\end{tikzcd}$$

Where the functors P and $\widetilde{P}$ have opcleavages $\{f_*, \psi_f, c_{fg}\}$ and $\{\widetilde{f}_*, \widetilde{\psi}_f, \widetilde{c}_{fg}\}$ respectively.

Let $(A, M, d) \in \mathcal{D}(A)$ be any object. Then

$\widetilde{f}_* \; I_A (A, M, d) = \widetilde{f}_* (A, M, d) = (B, M', d')$ by Prop (3.2)

$I_B \; f_* (A, M, d) = I_B (B, M', d') = (B, M', d')$ by Prop (2.2).

Hence, we have the following diagram commutative.

$$\begin{tikzcd}
\mathcal{D}(A) \arrow[r, "I_A"] \arrow[d, "f_*"'] & \mathcal{C}(A) \arrow[d, "\widetilde{f}_*"] \\
\mathcal{D}(B) \arrow[r, "I_B"] & \mathcal{C}(B)
\end{tikzcd}$$

It follows obviously that $\tilde{f}_* I_A = I_B f_*$.

For every $(A, M, d) \in \mathcal{D}(A)$, $f : A \longrightarrow B$ induces

$$I((\psi_f)_M) : I_A(A,M,d) \longrightarrow I_B\, f_*(A,M,d)$$

in $\mathcal{C}$. By Axiom (1) of opcleavage, there exists a unique morphism

$$(\eta_f)_M : \tilde{f}_* I_A\, (A, M, d) \longrightarrow I_B\, f_*(A, M, d)$$

making the following diagram commutative

$$\begin{tikzcd}
I_A\,(A, M, d) \arrow[r, "(\psi_f)_{IM}"] \arrow[dr, "I((\psi_f)_M)"'] & \tilde{f}_* I_A(A,M,d) \arrow[d, "(\eta_f)_M"] \\
& I_B\, f_*(A,M,d)
\end{tikzcd}$$

It is obvious that $(\eta_f)_M$ is the identity on

$$\tilde{f}_* I_A\, (A,M,d) = I_B\, f_*\, (A, M, d)$$

Therefore, we have $I\,(\psi_f) = \tilde{\psi}_f$.

It also follows obviously that $I\,(c_{fg}) = \tilde{c}_{fg}$.

Thus the functor I is opcleavage-preserving.