

CHAPTER - V

Let \mathcal{C} denote the category of all R -complexes and \mathcal{G} denote the category of all anticommutative graded algebras. Consider the forgetful functor

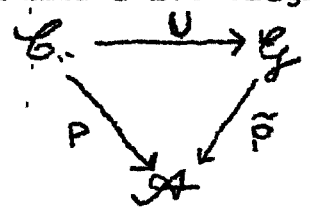
$$U : \mathcal{C} \longrightarrow \mathcal{G}$$

defined by $U(X,d) = X$ for all $(X,d) \in \mathcal{C}$

We claim :

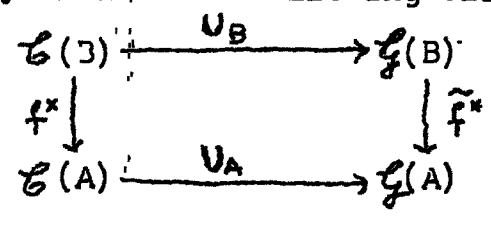
Theorem (5.1) : The functor $U : \mathcal{C} \rightarrow \mathcal{G}$ is cleavage preserving.

Proof : Consider a commutative diagram of functors



Where the functors P and \tilde{P} have cleavages $\{f^*, \Theta_f, d_{fg}\}$ and $\{\tilde{f}^*, \tilde{\Theta}_f, \tilde{d}_{fg}\}$ respectively. Let $U|_{\mathcal{C}(A)} = U_A$ for all $A \in \mathcal{A}$.

Let $(X,d) \in \mathcal{C}(B)$ be any object. Then $U_A f^*(X,d) = U_A(\bar{X}, \bar{d}) = \bar{X} = A \otimes_{\sum_{n \geq 1} X_n}$ by Prop (3.5). $\tilde{f}^* U_B(X,d) = \tilde{f}^* X = \bar{X} = A \otimes_{\sum_{n \geq 1} X_n}$ by Prop (4.5). Hence the following diagram commutes.



It follows obviously that $U_A f^* = \tilde{f}^* U_B$.

For every $(X; d) \in \mathcal{C}(B)$, $f: A \rightarrow B$ induces

$$U((\theta_f)_X) : U_A f^*(X, d) \longrightarrow U_B(X, d) \text{ in } \mathcal{C}$$

By Axiom (1) of cleavage, there exists a unique morphism

$$(\eta_f)_X : U_A f^*(X, d) \longrightarrow \tilde{f}^* U_B(X, d)$$

making the following diagram commutative :

$$\begin{array}{ccc} U_A f^*(X, d) & & \\ \downarrow (\eta_f)_X & \searrow U((\theta_f)_X) & \\ f^* U_B(X, d) & \xrightarrow{(\tilde{\theta}_f)_X} & U_B(X, d) \end{array}$$

It is obvious that $(\eta_f)_X$ is the identity on $U_A f^*(X, d) = \tilde{f}^* U_B(X, d)$.

Therefore, we have $U(\theta_f) = \tilde{\theta}_f$.

It also follows obviously that $U(d_{fg}) = \tilde{d}_{fg}$. Hence the functor U is cleavage - preserving.

Theorem (5.2) : The functor $U : \mathcal{C} \rightarrow \mathcal{C}$ is opcleavage-preserving.

Proof : Consider a commutative diagram of functors -

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{U} & \mathcal{C} \\ \downarrow P & & \downarrow \tilde{P} \\ \mathcal{A} & & \mathcal{A} \end{array}$$

Where the functors P and \tilde{P} have opcleavages $\{f_*, \psi_f, c_{fg}\}$ and $\{\tilde{f}_*, \tilde{\psi}_f, \tilde{c}_{fg}\}$ respectively.

Let $U|_{\mathcal{C}(A)} = U_A$ for all $A \in \mathcal{A}$.

Let $(X, d) \in \mathcal{C}(A)$ be any object. Then ~~\tilde{f}_*~~

$$\tilde{f}_* U_A(X, d) = \tilde{f}_* X' = X' \quad \text{by Prop (4.2).}$$

$$U_B f_* (X, d) = U_B (X', d') = X' \quad \text{by Prop (3.2).}$$

Hence, we have the following diagram commutative

$$\begin{array}{ccc} \mathcal{C}(A) & \xrightarrow{U_A} & \mathcal{C}(A) \\ f_* \downarrow & & \downarrow \tilde{f}_* \\ \mathcal{C}(B) & \xrightarrow{U_B} & \mathcal{C}(B) \end{array}$$

It follows obviously that $\tilde{f}_* U_A = U_B f_*$.

For every $(X, d) \in \mathcal{C}(A)$, $f : A \rightarrow B$ induces

$$U((\psi_f)_X) : U_A(X, d) \longrightarrow U_B f_* (X, d) \text{ in } \mathcal{C}. \text{ By}$$

Axiom (1) of opcleavage, there exists a unique morphism

$$(\eta_f)_X : \tilde{f}_* U_A(X, d) \longrightarrow U_B f_* (X, d)$$

making the following diagram commutative.

$$\begin{array}{ccc} U_A(X, d) & \xrightarrow{(\tilde{\psi}_f)_X} & \tilde{f}_* U_A(X, d) \\ & \searrow U((\psi_f)_X) & \downarrow (\eta_f)_X \\ & & U_B f_* (X, d) \end{array}$$

It is obvious that $(\eta_f)_X$ is the identity on

$$\tilde{f}_* U_A(X, d) = U_B f_* (X, d).$$

Therefore, we have $U(\psi_f) = \tilde{\psi}_f$.

It also follows obviously that $U(C_{fg}) = \tilde{C}_{fg}$.

Hence, the functor U is opcleavage-preserving.

Let \mathcal{D} denote the category of all R -derivation modules.

Consider the projection functor

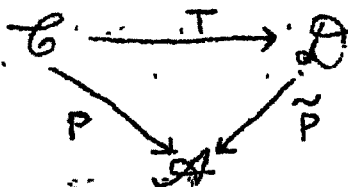
$$T : \mathcal{C} \longrightarrow \mathcal{D}$$

defined by $T(X, d) = (X_0, X_1, d_0)$ for all $(X, d) \in \mathcal{C}$

We claim :

Theorem (5.3) : The functor $T : \mathcal{C} \rightarrow \mathcal{D}$ is cleavage-preserving.

Proof : Consider a commutative diagram of functors



Where the functors P and \tilde{P} have cleavages $\{f^*, \theta_f, d_{fg}\}$ and $\{\tilde{f}^*, \tilde{\theta}_f, \tilde{d}_{fg}\}$ respectively.

Let $T|_{\mathcal{C}(A)} = T_A$ for all $A \in \mathcal{A}$

Let $(X, d) \in \mathcal{C}(B)$ be any object. Then

$$T_A f^*(X, d) = T_A(\bar{X}, \bar{d}) = (\bar{X}_0, \bar{X}_1, \bar{d}_0) = (A, \bar{X}_1, \bar{d}_0) \text{ by Prop (3.5)}$$

$$\tilde{f}^* T_B(X, d) = \tilde{f}^*(X_0, X_1, d_0) = (A, \bar{X}_1, \bar{d}_0) \text{ by Prop (2.5).}$$

Hence the following diagram commutes.

$$\begin{array}{ccc}
 \mathcal{C}(B) & \xrightarrow{T_B} & \mathcal{D}(B) \\
 f^* \downarrow & & \downarrow \tilde{f}^* \\
 \mathcal{C}(A) & \xrightarrow{T_A} & \mathcal{D}(A)
 \end{array}$$

It follows obviously that $T_A f^* = \tilde{f}^* T_B$.

For every $(X; d) \in \mathcal{C}(B)$, $f: A \rightarrow B$ induces

$$T((\theta_f)_X) : T_A f^*(X, d) \rightarrow T_B(X, d)$$

in \mathcal{D} . By Axiom (I) of cleavage, there exists a unique morphism

$$(\eta_f)_X : T_A f^*(X, d) \rightarrow \tilde{f}^* T_B(X, d)$$

making the following diagram commutative :

$$\begin{array}{ccc}
 T_A f^*(X, d) & & \\
 \downarrow (\eta_f)_X & \searrow T((\theta_f)_X) & \\
 f^* T_B(X, d) & \xrightarrow{(\tilde{\theta}_f)_{T_X}} & T_B(X, d)
 \end{array}$$

It is obvious that $(\eta_f)_X$ is the identity on

$$T_A f^*(X, d) = \tilde{f}^* T_B(X, d).$$

Therefore, we have $T(\theta_f) = \tilde{\theta}_f$.

It also follows obviously that $T(d_{fg}) = \tilde{d}_{fg}$. Thus the functor T is cleavage-preserving.

Theorem (5.4) : The functor $T : \mathcal{C} \rightarrow \mathcal{D}$ is opcleavage-preserving.

Proof: Consider a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{T} & \mathcal{D} \\ & \searrow P & \swarrow \tilde{P} \end{array}$$

Where the functors P and \tilde{P} have opcleavages $\{f_*, \psi_f, c_{fg}\}$ and $\{\tilde{f}_*, \tilde{\psi}_f, \tilde{c}_{fg}\}$ respectively.

Let $(X, d) \in \mathcal{C}(A)$ be any object. Then $\tilde{f}_* T_A(X, d) = \tilde{f}_*(A, X_1, d_0) = (B, X'_1, d'_0)$ by Prop (2.2). $T_B f_*(X, d) = T_B(X_1, d_1) = (B, X'_1, d'_0)$ by Prop (3.2).

Hence, we have the following diagram commutative.

$$\begin{array}{ccc} \mathcal{C}(A) & \xrightarrow{T_A} & \mathcal{D}(A) \\ f_* \downarrow & & \downarrow \tilde{f}_* \\ \mathcal{C}(B) & \xrightarrow{T_B} & \mathcal{D}(B) \end{array}$$

It follows obviously that $\tilde{f}_* T_A = T_B f_*$.

For every $(X, d) \in \mathcal{C}(A)$, $f : A \rightarrow B$ induces

$$T((\psi_f)_X) : T_A(X, d) \longrightarrow T_B f_*(X, d)$$

in \mathcal{D} . By Axiom (1) of opcleavage, there exists a unique morphism

$$(\eta_f)_X : \tilde{f}_* T_A(X, d) \longrightarrow T_B f_*(X, d)$$

making the following diagram commutative

$$\begin{array}{ccc}
 T_A(X, d) & \xrightarrow{(\tilde{\psi}_f)_X} & \tilde{f}_* T_A(X, d) \\
 & \searrow T((\psi_f)_X) & \downarrow (\eta_f)_X \\
 & & T_B f_* (X, d)
 \end{array}$$

It is obvious that $(\eta_f)_X$ is the identity on $\tilde{f}_* T_A(X, d) = T_B f_* (X, d)$.

Therefore, we have $T((\psi_f)_X) = \tilde{\psi}_f$.

It also follows obviously that $T(c_{fg}) = \tilde{c}_{fg}$.

Hence, the functor T is opcleavage - preserving.

By Remark (9) of Chapter II, it follows that every R -derivation module (A, M, d) can be considered as an R -complex (X, δ) where $X_0 = A$, $X_1 = M$, $X_n = 0$ for $n \geq 2$, and $\delta_0 = d$, $\delta_n = 0$ for $n \geq 1$.

Consider the inclusion functor

$$I : \mathcal{D} \rightarrow \mathcal{C}$$

defined as above. We claim :

Theorem (5.5) : The functor $I : \mathcal{D} \rightarrow \mathcal{C}$ is cleavage-preserving.

Proof : Consider a commutative diagram of functors -

$$\begin{array}{ccc}
 \mathcal{D} & \xrightarrow{I} & \mathcal{C} \\
 \downarrow P & & \uparrow \tilde{P} \\
 \mathcal{A} & & \mathcal{A}
 \end{array}$$

Where the functors P and \tilde{P} have cleavages $\{f^*, \theta_f, d_{fg}\}$ and $\{\tilde{f}^*, \tilde{\theta}_f, \tilde{d}_{fg}\}$ respectively.

Let $I \mid \mathcal{D}(A) = I_A$ for all $A \in \mathcal{A}$

Let $(B, M, d) \in \mathcal{D}(B)$ be any object. Then

$$I_A f^*(B, M, d) = I_A(A, \bar{M}, \bar{d}) = (A, \bar{M}, \bar{d}) \quad \text{by Prop (2.5)}$$

$$\tilde{f}^* I_B(B, M, d) = \tilde{f}^*(B, M, d) = (A, \bar{M}, \bar{d}) \quad \text{by Pro. (3.5)}$$

Hence the following diagram commutes.

$$\begin{array}{ccc} \mathcal{D}(B) & \xrightarrow{I_B} & \mathcal{C}(B) \\ f^* \downarrow & & \downarrow \tilde{f}^* \\ \mathcal{D}(A) & \xrightarrow{I_A} & \mathcal{C}(A) \end{array}$$

It follows obviously that $I_A f^* = \tilde{f}^* I_B$.

For every $(B, M, d) \in \mathcal{D}(B)$, $f : A \rightarrow B$ induces

$$I((\theta_f)_M) : I_A f^*(B, M, d) \longrightarrow I_B(B, M, d)$$

in \mathcal{C} . By Axiom (1) of cleavage, there exists a unique morphism,

$$(\eta_f)_M : I_A f^*(B, M, d) \longrightarrow \tilde{f}^* I_B(B, M, d)$$

making the following diagram commutative

$$\begin{array}{ccc} I_A f^*(B, M, d) & & \\ \downarrow (\eta_f)_M & \searrow I((\theta_f)_M) & \\ \tilde{f}^* I_B(B, M, d) & \xrightarrow{(\tilde{\theta}_f)_M} & I_B(B, M, d) \end{array}$$

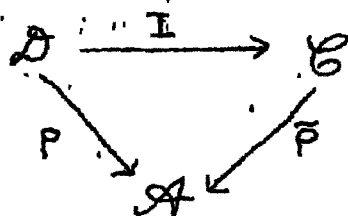
It is obvious that $(\eta_f)_M$ is the identity on $I_A \cdot f^*(B, M, d) = f^* I_B(B, M, d)$.

Therefore, we have $I(\Theta_f) = \tilde{\Theta}_f$.

It also follows obviously that $I(d_{fg}) = \tilde{d}_{fg}$. Thus the functor I is cleavage-preserving.

Theorem (5.6) : The functor $I : \mathcal{D} \rightarrow \mathcal{C}$ is opcleavage-preserving.

Proof : Consider a commutative diagram of functors



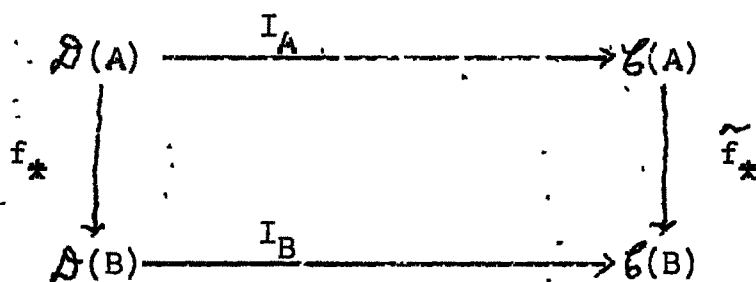
Where the functors P and \tilde{P} have opcleavages $\{f_*, \psi_f, c_{fg}\}$ and $\{\tilde{f}_*, \tilde{\psi}_f, \tilde{c}_{fg}\}$ respectively.

Let $(A, M, d) \in \mathcal{D}(A)$ be any object. Then

$$f_* I_A(A, M, d) = \tilde{f}_*(A, M, d) = (B, M', d') \text{ by Prop. (3.2)}$$

$$I_B f_*(A, M, d) = I_B(B, M', d') = (B, M', d') \text{ by Prop. (2.2).}$$

Hence, we have the following diagram commutative.



It follows obviously that $\tilde{f}_* I_A = I_B f_*$.

For every $(A, M, d) \in \mathcal{D}(A)$, $f : A \rightarrow B$ induces

$$I((\psi_f)_M) : I_A(A, M, d) \longrightarrow I_B f_*(A, M, d)$$

in \mathcal{C} . By Axiom (1) of opcleavage, there exists a unique morphism

$$(\eta_f)_M : \tilde{f}_* I_A(A, M, d) \longrightarrow I_B f_*(A, M, d)$$

making the following diagram commutative

$$\begin{array}{ccc} I_A(A, M, d) & \xrightarrow{(\psi_f)_{IM}} & \tilde{f}_* I_A(A, M, d) \\ & \searrow \tau((\psi_f)_M) & \downarrow (\eta_f)_M \\ & & I_B f_*(A, M, d) \end{array}$$

It is obvious that $(\eta_f)_M$ is the identity on

$$\tilde{f}_* I_A(A, M, d) = I_B f_*(A, M, d)$$

Therefore, we have $I(\psi_f) = \tilde{\psi}_f$.

It also follows obviously that $I(c_{fg}) = \tilde{c}_{fg}$.

Thus the functor I is opcleavage-preserving.