CHAPTER - I

CATEGORIES AND COMPLEXES

PRELIMINARIES :

This chapter essentially consists of all the basic definitions and results which will be needed in the ensuing chapters.

Definition (1.1) : A category $\mathcal{A}$ consists of

(a) class of objects;

(b) for every ordered pair of objects A, B $\in \mathcal{A}$, a set $\mathcal{A}[A, B]$ of morphisms with domain A and codomain B.

If $f \in \mathcal{A}[A,B]$ we write $f:A \to B$ or $A \xrightarrow{f} B$.

(c) for every ordered triple of objects A, B, C $\in \mathcal{A}$ a function $\mathcal{A}[B,C] \times \mathcal{A}[A,B] \to \mathcal{A}[A,C]$ called composition delined as $(g,f) \mapsto g\,f$. satisfying the following two axioms :

i) Associativity : $h\,(g\,f) = (h\,g)\,f$ whenever the compositions make sense.

ii) Existence of identities : For each $A \in \mathcal{A}$ there exists $I_A \in \mathcal{A}[A,A]$ such that $f\,I_A = f$ and $I_A\,g = g$ whenever the compositions make sense.

We shall denote a category and the class of objects by the same letter.

<u>Definition (1.2)</u> : If the morphisms $f: A \longrightarrow B$ and $g : B \longrightarrow A$ are such that $g\ f = I_A$ then g is called a <u>left inverse of</u> f and f is called the <u>right inverse</u> of g.

<u>Definition (1.3)</u> : If the morphisms $f : A \longrightarrow B$ and $g: B \longrightarrow A$ are such that $g\ f = I_A$ and $f\ g = I_B$ then g is called the <u>inverse</u> of f.

<u>Definition (1.4)</u> : A morphism $f: A \longrightarrow B$ is called an <u>isomorphism</u> if it has the inverse.

<u>Definition (1.5)</u> : When such an isomorphism exists in $\mathcal{A}[A,B]$, we say that A is isomorphic to B.

<u>Definition (1.6)</u> : A morphism $f: A \longrightarrow B$ is called a monomorphism if $f\ g = f\ h$ implies $g = h$ for all g, h $\mathcal{A}[C,A]$ for all $C \in \mathcal{A}$.

<u>Definition (1.7)</u> : If a monomorphism exists in $\mathcal{A}[A, B]$ then A is called a <u>subobject</u> of B.

<u>Definition (1.8)</u> : A morphism $f : A \longrightarrow B$ is called an epimorphism if $g\ f = h\ f$ implies $g = h$ for all $g, h \in \mathcal{A}[B,D]$ for all $D \in \mathcal{A}$.

<u>Definition (1.9)</u> : An object $U \in \mathcal{A}$ is called an <u>initial object</u> of $\mathcal{A}$ if the set $\mathcal{A}[U,A]$ contains precisely one morphism for each $A \in \mathcal{A}$.

In any category, the initial object (if it exists) is unique upto isomorphism.

<u>Definition (1.10)</u> : Let $\mathcal{A}$ and $\mathcal{B}$ be categories. A (covariant) <u>functor</u> $T : \mathcal{A} \to \mathcal{B}$ consists of

(a) an object function which assigns an object $T(A) \in \mathcal{B}$ to each object $A \in \mathcal{A}$; and

(b) a morphism function which assigns a morphism $T(f) : T(A) \longrightarrow T(B)$ in $\mathcal{B}$ to each morphism $f: A \longrightarrow B$ in $\mathcal{A}$ in such a way that :

i) For each $A \in \mathcal{A}$, we have $T(I_A) = I_{T(A)}$.

ii) If $g\ f$ is defined in $\mathcal{A}$ then $T(g\ f) = T(g)\ T(f)$.

<u>Definition (1.11)</u> : Given two functors $S, T : \mathcal{A} \to \mathcal{B}$, <u>a natural transformation</u> $\eta : S \longrightarrow T$ is a function which assigns to each object $X \in \mathcal{A}$, a morphism $\eta_X : S(X) \longrightarrow T(X)$ in $\mathcal{B}$, in such a way that every morphism $f: A \longrightarrow B$ in $\mathcal{A}$ yields a commutative diagram

$$\begin{array}{ccc} S(A) & \xrightarrow{\eta_A} & T(A) \\ {\scriptstyle S(f)}\downarrow & & \downarrow{\scriptstyle T(f)} \\ S(B) & \xrightarrow{\eta_B} & T(B) \end{array}$$

η_X are called the components of the natural transformation η.

If, η_X is an isomorphism for each $X \in \mathcal{A}$, then η is called a natural equivalence.

Definition (1.12) : A category $\mathcal{A}'$ is said to be a subcategory of the category $\mathcal{A}$ if the following conditions are satisfied :

(a) $\mathcal{A}' \subseteq \mathcal{A}$.

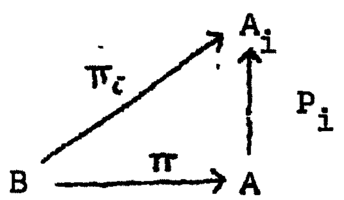
(b) $\mathcal{A}'[A,B] \subseteq \mathcal{A}[A,B]$ for all $(A,B) \in \mathcal{A}' \times \mathcal{A}'$.

(c) the composition of any two morphisms in $\mathcal{A}'$ is the same as their composition in $\mathcal{A}$.

(d) I_X is the same in $\mathcal{A}'$ as in $\mathcal{A}$ for all $X \in \mathcal{A}'$

Definition (1.13) : A subcategory $\mathcal{A}'$ of $\mathcal{A}$ is called full subcategory if $\mathcal{A}'[A,B] = \mathcal{A}[A,B]$ for all $(A,B) \in \mathcal{A}' \times \mathcal{A}'$

Definition (1.14) : Let $\{A_i\}_{i \in I}$ be a set of objects in an arbitrary category $\mathcal{A}$. A product for this family is a family of morphisms $\{P_i : A \longrightarrow A_i\}$ with the property that for any family $\{\pi_i : B \longrightarrow A_i\}$, there is a unique morphism $\pi : B \longrightarrow A$ such that the following diagram commutes for every $i \in I$.



The object A will be denoted by $\prod_{i \in I} A_i$.

Definition (1.15) : CLEAVAGE

Let $\mathcal{E}$ and $\mathcal{B}$ be categories and $P : \mathcal{E} \to \mathcal{B}$ be a functor. For each $B \in \mathcal{B}$, let $\mathcal{E}(B) = P^{-1}(B)$ denote the (possibly empty) fibre of P over B, where the fibre $\mathcal{E}(B)$ is a subcategory of $\mathcal{E}$ consisting of those objects of $\mathcal{E}$ which are mapped onto B by the functor P and those morphisms which are mapped onto I_B by P. Let $J_B : \mathcal{E}(B) \to \mathcal{E}$ be the inclusion functor. A cleavage consists of functors $f^* : \mathcal{E}(B) \longrightarrow \mathcal{E}(B')$ for each morphism $f : B' \longrightarrow B$ in $\mathcal{B}$, together with natural transformations $\theta_f : J_{B'} \, f^* \longrightarrow J_B$ satisfying two axioms.

Axiom (1) : $P(\theta_f) = f$ and if $\varphi : E' \longrightarrow E$ in $\mathcal{E}$ satisfies $P(\varphi) = f$, there exists a unique $\varphi' : E' \longrightarrow f^*E$ in $\mathcal{E}(B')$ such that $(\theta_f)_E \; \varphi' = \varphi$.

$$\begin{array}{ccc} E' & & \\ {\scriptstyle \varphi'}\downarrow & \searrow^{\varphi} & \\ f^*E & \xrightarrow{(\theta f)_E} & E \\ B' & \xrightarrow{\quad f \quad} & B \end{array}$$

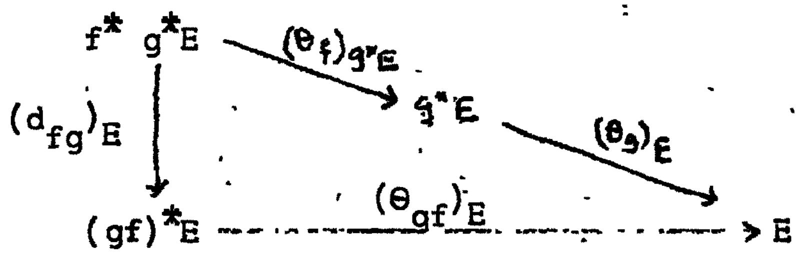
[We usually omit the subscript E from $(\theta_f)_E$ when it is clear which component of the natural transformation is relevant.]

To state the second axiom, consider the composition $B'' \xrightarrow{f} B' \xrightarrow{g} B$ in $\mathcal{B}$. Then for each $E \in \mathcal{E}(B)$ there is a uniquely determined morphism

$$(d_{fg})_E : f^* g^* E \longrightarrow (g\ f)^* E \text{ in } \mathcal{E}(B'') \text{ such that}$$

$$(\Theta_{gf})_E \quad (d_{fg})_E = (\Theta_g)_E (\Theta_f)_{g^* E}$$

making the following diagram commutative.



It is easily checked that $(d_{fg})_E$ are the components of a natural transformation $d_{fg} : f^* g^* \longrightarrow (g\, f)^*$

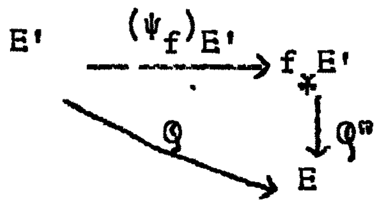
Axiom (2) : Each d_{fg} is a natural equivalence.

Definition (1.16) : A cleavage is called normalized if $(I_B)^* = I_{\mathcal{E}(B)}$ for all $B \in \mathcal{B}$.

Definition (1.17) : A cleavage is called split if each d_{fg} is the identity natural transformation.

Definition (1.18) : An opposite cleavage or opcleavage consists of functors $f_* : \mathcal{E}(B') \longrightarrow \mathcal{E}(B)$ for each morphism $f : B' \longrightarrow B$ in $\mathcal{B}$ together with natural transformations $\psi_f : J_{B'} \longrightarrow J_B f_*$ satisfying two axioms :

Axiom (1) : $P(\psi_f) = f$ and if $\varphi : E' \longrightarrow E$ in $\mathcal{E}$ satisfies $P(\varphi) = f$, there exists a unique $\varphi^{u} : f_* E' \longrightarrow E$ in $\mathcal{E}(B)$ such that $\varphi^{u} (\psi_f)_{E'} = \varphi$.



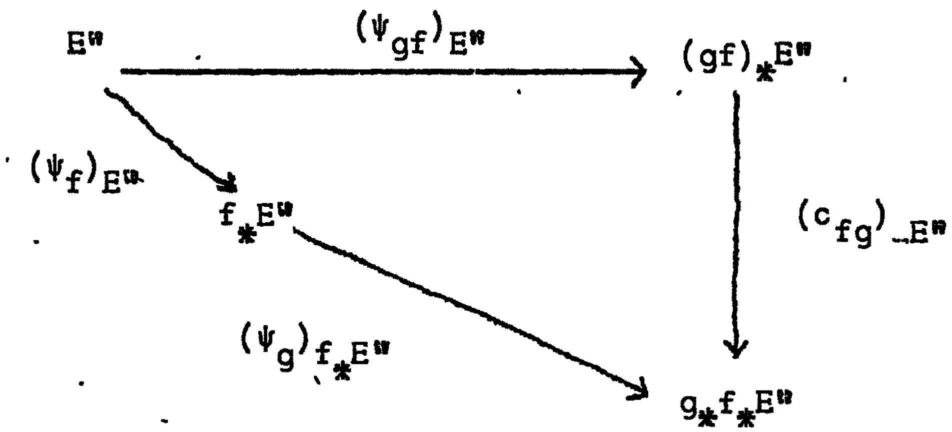
To state the second axiom consider the composition $B'' \xrightarrow{f} B' \xrightarrow{g} B$ in $\mathcal{B}$. Then for each $E'' \in \mathcal{E}(B'')$ there is a uniquely determined morphism

$$(c_{fg})_{E''} : (gf)_* E'' \longrightarrow g_* f_* E''$$

in $\mathcal{E}(B)$ such that

$$(c_{fg})_{E''} (\psi_{gf})_{E''} = (\psi_g)_{f_*E''} \cdot (\psi_f)_{E''}$$

making the following diagram commutative.



It is easily checked that $(c_{fg})_{E''}$ are the components of a natural transformation

$$c_{fg} : (gf)_* \longrightarrow g_* f_*$$

Axiam (2) : Each c_{fg} is a natural equivalence.

<u>Definition</u> (1.19) : Consider a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ T\ } & \tilde{\mathcal{E}} \\ & {\scriptstyle P}\searrow \quad \swarrow{\scriptstyle \tilde{P}} & \\ & \mathcal{B} & \end{array}$$

where P and $\tilde{P}$ have cleavages $\{f^*, \theta_f, d_{fg}\}$ and $\{\tilde{f}^*, \tilde{\theta}_f, \tilde{d}_{fg}\}$ respectively. Then $T | \mathcal{E}(B) = T_B$ for all $B \in \mathcal{B}$ and if $f: B' \longrightarrow B$ is a morphism in $\mathcal{B}$, then there is a unique natural transformation

$$\eta_f : T_{B'} f^* \longrightarrow \tilde{f}^* T_B$$

such that $\tilde{P}(\eta_f) = I_{B'}$ and $T(\theta_f) = \tilde{\theta}_f \eta_f$. ______ (I)

These transformations satisfy a complicated relation

$$\tilde{d}_{fg} \; f^*[\eta_g] \; (\eta_f)_{g^*E} = \eta_{gf} \;\; T_{B''} \; [(d_{fg})_E]$$

for $B'' \xrightarrow{f} B' \xrightarrow{g} B$ in $\mathcal{B}$ ______ (II)

If η is identity for all f, then the functor T is called <u>cleavage-preserving</u>. i.e. if $T_{B'} f^* = \tilde{f}^* T_B$

then by (I) we have $T(\theta_f) = \tilde{\theta}_f$

and by (II) we have $T(d_{fg}) = \tilde{d}_{fg}$.

<u>Definition (1.20)</u> : Consider a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\ T\ } & \tilde{\mathcal{E}} \\ & {\scriptstyle P}\searrow \quad \swarrow{\scriptstyle \tilde{P}} & \\ & \mathcal{B} & \end{array}$$

Where P and $\tilde{P}$ have opcleavages $\{f_*, \psi_f, c_{fg}\}$ and $\{\tilde{f}_*, \tilde{\psi}_f, \tilde{c}_{fg}\}$ respectively. Then $T \mid \mathcal{E}(B) = T_B$ for all $B \in \mathcal{B}$ and if $f : B' \longrightarrow B$ is a morphism in $\mathcal{B}$, then there is a unique natural transformation

$$\eta_f : \tilde{f}_* \quad T_{B'} \longrightarrow T_B\, f_*$$

such that $\tilde{P}(\eta_f) = I_{B'}$ and $T(\psi_f) = \eta_f\, \tilde{\psi}_f$ ——— (I)

These transformations satisfy a complicated relation

$$(\eta_g)_{f_* E''} \quad \tilde{g}_*[\eta_f] \;\; \tilde{c}_{fg} = T_B\, [(c_{fg})_{E''}]\; \eta_{gf} \qquad \text{——(II)}$$

for $B'' \xrightarrow{f} B' \xrightarrow{g} B$ in $\mathcal{B}$.

If η is identity for all f, then the functor T is called <u>opcleavage-preserving</u>.

i.e. if $\tilde{f}_* \;\; T_{B'} = T_B\, f_*$

then by (I) we have $T(\psi_f) = \tilde{\psi}_f$ and by (II) $T(c_{fg}) = \tilde{c}_{fg}$

<u>Convention</u> : By a ring R we shall always mean a commutative unitary ring and by an R-algebra A we mean a commutative unitary algebra over R. $\mathcal{A}$ Will always denote the category of unitary commutative R-algebras and unitary algebra homomorphisms.

<u>Definition (1.21)</u> : An algebra $X = \sum_{n \geq o} X_n$ (dir), X_n being the R-submodules of X, is called an <u>anti-commutative graded algebra</u> if;

i) $X_n \cdot X_m \subseteq X_{n+m}$ for all n, m $\geq$ o;

ii) for all n, m $\geq$ o, $x \in X_n$, $x' \in X_m$ implies $x.x' = (-1)^{nm}\, x'.x$;

iii) x in X_n, n being odd implies $x^2 = 0$.

If $x \in X_n$ then x is said to be homogeneous of degree n. If $x \in X$, then x is represented in a unique way in the form $x = \sum_{n \geq 0} x_n$ where x_n is homogeneous of degree n.

Definition (1.22) : A subalgebra Y of an anticommutative graded algebra X is called homogeneous subalgebra if Y has a set of module generators composed of homogeneous elements.

Remark (1.1) : Any commutative R-algebra can be considered as an anticommutative graded algebra with degree 0 for every element.

Remark (1.2) : If Y is a homogeneous subalgebra of the anticommutative graded algebra X, then Y is an anticommutative graded algebra equipped with the gradation induced by that of X.

Definition (1.23) : Let $X = \sum_{n \geq 0} X_n$ and $Y = \sum_{n \geq 0} Y_n$ be anticommutative graded R-algebras. Then the R-algebra hemomorphism $\varphi : X \longrightarrow Y$ is called a graded algebra homomorphism if for each $n \geq 0$ $\varphi(X_n) \subseteq Y_n$.

Proposition (1.1) : Let $(X_\alpha)_{\alpha \in I}$ be a family of anticommutative graded R-algebras such that for each $\alpha \in I$, $X_\alpha = \sum_{n \geq 0} X_{\alpha,n}$ (dir) is the gradation of X_α. Then $\prod_\alpha X_\alpha = \sum_{n \geq 0} (\prod_\alpha X_{\alpha,n})$ is an anticommutative graded R-algebra.

Proof : We know that $\pi_\alpha X_\alpha$ is an R-algebra and that for each $n \geqslant 0$, $\pi_\alpha X_{\alpha,n}$ is an R-submodule of $\pi_\alpha X_\alpha$. Set $\mathbb{P}_\alpha X_\alpha = \sum_{n \geqslant 0} (\pi_\alpha X_{\alpha,n})$

This sum is direct; and $\mathbb{P}_\alpha X_\alpha$ is a subalgebra of $\pi_\alpha X_\alpha$. We claim that $\mathbb{P}_\alpha X_\alpha$ is anticommutative. For this take arbitrary homogeneous elements $x = (x_{\alpha,n})_\alpha$ and $y = (y_{\alpha,m})_\alpha$ of respective degrees n and m in $\mathbb{P}_\alpha X_\alpha$. Then $xy \in \pi_\alpha X_{\alpha,n+m}$.
Since $xy = (x_{\alpha,n} \cdot y_{\alpha,m})_\alpha = ((-1)^{nm} y_{\alpha,m} \cdot x_{\alpha,n})_\alpha = (-1)^{nm} yx$.
Similarly $x^2 = 0$ if n is odd.
Hence $\mathbb{P}_\alpha X_\alpha$ is an anticommutative graded R-algebra.

Definition (1.24) : For a family $(X_\alpha)_{\alpha \in I}$ of anticommutative graded R-algebras, the R-algebra $\mathbb{P}_\alpha X_\alpha = \sum_{n \geqslant 0} (\pi_\alpha X_{\alpha,n})$ is called the "product" of the family $(X_\alpha)_{\alpha \in I}$.

Convention : $\mathcal{G}$ will always denote the category of all anticommutative graded algebras with morphisms the graded algebra homomorphisms.

Definition (1.25) : Let X and Y be anticommutative graded algebras. For any $m \geqslant 0$, a homogeneous algebra homomorphism of degree m is an algebra homomorphism $\varphi : X \longrightarrow Y$ such that $\varphi(X_n) \subseteq Y_{n+m}$ for all $n \geqslant 0$.

Definition (1.26) : Let $X = \sum_{n \geqslant 0} X_n$ be an anticommutative graded algebra. By an R-derivation of degree 1 of X we mean an R-linear mapping $d : X \longrightarrow X$ homogeneous of degree 1 such that $d(xy) = dx \cdot y + (-1)^m x \cdot dy$ where $x \in X_m$.

<u>Definition (1.27)</u> : An <u>R-complex</u> is a pair (X,d) where X is an anticommutative graded R-algebra and $d:X \rightarrow X$ is an R-derivation of degree 1 of X such that $d\,d = 0$.

<u>Definition (1.28)</u> :: An R-complex (Y,δ) is called an <u>R-subcomplex</u> of an R-complex (X,d) if

i) Y is homogeneous subalgebra of X such that $dY \subseteq Y$

ii) the restriction of d to Y is δ.

<u>Remark (1.3)</u> : If (X, d) is an R-complex with $X_n = 0$ for $n \geqslant 1$, then $d = 0$.

<u>Remark (1.4)</u> : If d_n denotes the restriction of d to X_n, then we have a sequence

$$X_o \xrightarrow{d_o} X_1 \xrightarrow{d_1} X_2 \xrightarrow{d_2} \cdots\cdots$$

of modules and derivations such that $d_n\, d_{n-1} = 0$ for all $n \geqslant 1$.

<u>Remark (1.5)</u> : The intersection of a collection of R-subcomplexes of an R-complex (X, d) is again an R-subcomplex of (X,d).

<u>Remarks (1.6)</u> : Let (X,d) be an R-complex and S a subset of X. Let $\{(X_\alpha,d_\alpha)\}_{\alpha \in I}$ be a collection of all R-subcomplexes of (X,d) such that $S \subseteq X_\alpha$ for each $\alpha \in I$. Then the intersection of $\{(X_\alpha,d_\alpha)\}_{\alpha \in I}$ is again an R-subcomplex (Y,δ) of (X,d) such that $S \subseteq Y$. Moreover, there does not exist a proper R-subcomplex (Z,∂) of (Y,δ) such that $S \subseteq Z$. Then (Y,δ) is called the <u>R-complex generated by S</u>.

Definition (1.29) : Let (X,d) and (Y,δ) be R-complexes. A complex homomorphism $f: (X,d) \longrightarrow (Y,\delta)$ is a graded algebra homomorphism $f : X \longrightarrow Y$ such that $f\, d = \delta\, f$.

Definition (1.30) : A complex homomorphism $f:(x,d) \longrightarrow (Y,\delta)$ is called an isomorphism if there exists a complex homomorphism $g : (Y,\delta) \longrightarrow (X,d)$ such that $g\, f$ is identity on (X,d), and $f\, g$ is identity on (Y,δ).

Proposition (1.2) :

A collection of all R-complexes together with complex homomorphisms forms a category $\mathscr{C}$

Proof : Let $\mathscr{C}$ denote the collection of all R-complexes. Then $\mathscr{C}$ is nonempty, because the complex (X,d) with X_0 = some R-algebra and $X_n = 0$ for $n \geqslant 1$, is in $\mathscr{C}$. Also for every (X,d) in $\mathscr{C}$, the identity mapping $I:X \longrightarrow X$ is trivially a complex homomorphism $I:(X,d) \longrightarrow (X,d)$. Let (X,d), (Y,δ) and (Z,∂) be R-complexes and let $f: (X,d) \longrightarrow (Y,\delta)$ and $g:(Y,\delta) \longrightarrow (Z,\partial)$ be any complex homomorphisms.

Then $g.f : (X,d) \longrightarrow (Z,\partial)$ is a complex homomorphism because $g.f\, d = g.\delta.f = \partial\, g.f$. Hence $\mathscr{C}$ is a category.

Proposition (1.3) : Let $\{(X_\alpha, d_\alpha)\}_{\alpha \in I}$ be a family of R-complexes. Then $(\underset{\alpha}{\Pi\!\!P} X_\alpha, d)$ is an R-complex where d is the restriction of $(d_\alpha)_\alpha : \underset{\alpha}{\pi} X_\alpha \longrightarrow \underset{\alpha}{\pi} X_\alpha$ to $\underset{\alpha}{\Pi\!\!P} X_\alpha$.

<u>Proof</u> : We know that $\prod_{\alpha} X_\alpha$ is an anticommutative graded R-algebra. We claim that $d: \prod_{\alpha} X_\alpha \longrightarrow \prod_{\alpha} X_\alpha$ is a derivation of degree 1 of $\prod_{\alpha} X_\alpha$ such that $d\,d = 0$. Recall that

$\prod_{\alpha} X_\alpha = \sum_{n \geq 0} \pi_\alpha X_{\alpha,n}$; and $(d_\alpha)_\alpha : \pi_\alpha X_\alpha \longrightarrow \pi_\alpha X_\alpha$ given by $(d_\alpha)_\alpha((x_\alpha)_\alpha) = (d_\alpha x_\alpha)_\alpha$ is R-Linear. Let $x = (x_{\alpha,n_1})_\alpha + (x_{\alpha,n_2})_\alpha + \cdots + (x_{\alpha,n_k})_\alpha$ be arbitrary element of $\prod X_\alpha$. – ( )

Then $d(x) = (d_\alpha)_\alpha ((x_{\alpha,n_1})_\alpha + \cdots + (x_{\alpha,n_k})_\alpha)$

$$= (d_\alpha x_{\alpha,n_1})_\alpha + \cdots + (d_\alpha x_{\alpha,n_k})_\alpha \in \prod_{\alpha} X_\alpha.$$

Since each d_α is of degree 1, d is of degree 1. Now let $x = (x_{\alpha,m})_\alpha$ and $y = (y_{\alpha,n})_\alpha$ be homogeneous elements of respective degrees m and n in $\prod_{\alpha} X_\alpha$. Then

$$d\,(x.y) = (d_\alpha (x_{\alpha,m}.y_{\alpha,n}))_\alpha$$

$$= (d_\alpha x_{\alpha,m}.y_{\alpha,n} + (-1)^m x_{\alpha,m}.dy_{\alpha,n})_\alpha$$

$$= dx.y + (-1)^m x.dy.$$

Moreover since $d_\alpha . d_\alpha = 0$ for each $\alpha \in I$ we have $d\,d = 0$. Therefore $(\prod_{\alpha} X_\alpha, d)$ is an R-complex.

<u>Definition (1.31)</u> : For a family $\{(X_\alpha, d_\alpha)\}_{\alpha \in I}$ of R-complexes, the R-complex $(\prod_{\alpha} X_\alpha, d)$ is called the "product" of the above family of R-complexes in the category

<u>Definition (1.32)</u> : Let A be a commutative unitary R-algebra. An R-complex (X,d) is called a <u>complex over A</u> or an <u>A-complex</u> if $X_o = A$.

<u>Definition (1.33)</u> : An R-subcomplex (Y,δ) of an A-complex (X,d) is called an <u>A-subcomplex</u> of (X,d) if $Y_o = A$.

<u>Remark (1.7)</u> : If (X,d) is an A-complex then X is an anticommutative graded A-algebra.

<u>Remark (1.8)</u> : Since any R-algebra A can be considered as an anticommutative graded A-algebra with $A_o = A$ and $A_n = 0$ for $n \geqslant 1$, we have that A together with the derivation d such that $d = 0$ is a complex over A.

<u>Remark (1.9)</u> : Let $f : A \rightarrow B$ be an R-algebra homomorphism. Then every B-complex (X,d) gives rise to an A-complex as follows : Define the scaler multiplication by the elements of A on X as follows : $ax = f(a)x$ for $a \in A$ and $x \in X$. Then X becomes an A-algebra. Since each X_n becomes an A-module with respect to this scaler multiplication, X becomes an anticommutative graded A-algebra. $\bar{X} = A + \sum_{n \geqslant 1} X_n$ is an anticommutative graded A-algebra. It is easy to check that $d\, f : A \rightarrow X_1$ is an R-derivation. Define $\bar{d} : \bar{X} \rightarrow \bar{X}$ as $\bar{d}_o = d\, f$ and $\bar{d}_n = d_n$ for $n \geqslant 1$. Then $(\bar{X}, \bar{d})$ is an A-complex.

<u>Remark (1.10)</u> : Every A-complex (X,d) contains an A-subcomplex (Y,δ) which is generated by A. In this case Y is the anticommutative graded algebra generated by the set dA as an A-algebra.

Definition (1.34) : An A-complex (X,d) is called simple if it does not contain any proper A-subcomplex.

Remark (1.11) : An A-complex (X,d) is simple if and only if X is generated by dA as an A-algebra.

Definition (1.35) : Let (X,d) and (Y,δ) be A-complexes. Then an R-complex homomorphism $\varphi : (X,d) \longrightarrow (Y,\delta)$ is called a complex homomorphism over A or A-complex homomorphism if φ maps A identically.

Remark (1.12) : An A-complex homomorphism is A-linear.

Proposition (1.4) : A collection $\mathscr{C}(A)$ of all A-complexes and A-complex homomorphisms, forms a category.

Remark (1.13) : $\mathscr{C}(A)$ is a subcategory of $\mathscr{C}$ consisting of fewer objects and fewer morphisms. $\mathscr{C}(A)$ is not full subcategory of $\mathscr{C}$.

Remark (1.14) : Let $P_\alpha : \underset{\alpha}{\mathbb{P}} X_\alpha \longrightarrow X_\alpha$ be the restriction of the natural projection $\pi_\alpha : \underset{\alpha}{\pi} X_\alpha \longrightarrow X_\alpha$ to $\underset{\alpha}{\mathbb{P}} X_\alpha$. Then P_α is also a complex homomorphism. From here onwards we shall denote P_α by π_α itself.

Remark (1.15) : Suppose for each $\alpha \in I$, (X_α, d_α) is a complex over A_α. Then $(\underset{\alpha}{\mathbb{P}} X_\alpha, d)$ is a complex over $\underset{\alpha}{\pi} A_\alpha$. If for each $\alpha \in I$, (X_α, d_α) is a complex over A, then define

$$\bar{A} = \{(a_\alpha)_\alpha \mid a_\alpha \in A_\alpha,\ a_\alpha = a \text{ for all } \alpha \in I\}$$

Clearly $\bar{A}$ is isomorphic to A. Now take the sum $\bar{A}+\sum_{n\geqslant 1}\prod_\alpha X_{\alpha,n}$ inside $\prod_\alpha X_\alpha$. Then clearly $(\bar{A}+\sum_{n\geqslant 1}\prod_\alpha X_{\alpha,n}, d)$ is an A-complex.

The A-complex $(\bar{A} + \sum_{n\geqslant 1}\prod_\alpha X_{\alpha,n} d)$ is the "product" of the family $\{(X_\alpha, d_\alpha)\}_{\alpha \in I}$ of A-complexes in the category $\mathcal{C}(A)$.

Remark (1.16) : Consider the restriction of

$$\pi_\alpha : \prod_\alpha X_\alpha \longrightarrow X_\alpha \quad \text{to } \bar{A} + \sum_{n\geqslant 1}\prod_\alpha X_{\alpha,n}.$$

Let this restriction be also denoted by π_α. Then

$$\pi_\alpha : (\bar{A} + \sum_{n\geqslant 1}\prod_\alpha X_{\alpha,n}, d) \longrightarrow (X_\alpha, d_\alpha) \quad \text{is a complex}$$

homomorphism over A.