CHAPTER - II

CLEAVAGES AND OPCLEAVAGES IN THE CATEGORY OF R-DERIVATION MODULES

Let R be a commutative ring with unity. Unless stated otherwise, by an algebra we mean a commutative unitary R-algebra. In the following, A, B, C will denote commutative unitary R-algebras and $f : A \longrightarrow B$, $g : B \longrightarrow C$ will denote R-algebra homomorphisms.

Definition (2.1) : An R-derivation module is an ordered triple (A, M, d) where A is a commutative unitary R-algebra. M is a unitary A-module and $d : A \longrightarrow M$ is an R- derivation.

Definition (2.2) : A derivation module (A, N, δ) is said to be a derivation A-submodule of a derivation module (A,M,d) if N is an A-submodule of M and d restricted to N is δ.

Remark (1) : Let (A, M, d) be an R-derivation module and $\varphi : M \longrightarrow N$ is an A - module homomorphism of M into another A - module N. Then $\varphi.d : A \longrightarrow N$ is an R - derivation. This gives a derivation module (A, N, φ d).

Definition (2.3) : A derivation module (A, M, d) is called simple if it does not contain a proper derivation A-submodule.

Remark (2) : Let (A, M, d) be a derivation module. Let N be the A-submodule of M generated by dA. Then (A, N, d) is a derivation A-submodule of the derivation module (A, M, d).

Remark (3) : A derivation module (A, M, d) is simple if and only if M is generated by dA as an A-module.

Remark (4) : Every derivation module (A, M, d) contains a simple derivation A-submodule (A, N, δ) and $|N| \leq |A| \aleph_0$.

Definition (2.4) : Let (A, M, d) and (B, N, δ) be two R-derivation modules. Then a derivation module homomorphism $\varphi : (A, M, d) \longrightarrow (B, N, \delta)$ is an ordered pair (φ_0, φ_1) where $\varphi_0 : A \longrightarrow B$ is an R-algebra homomorphism and $\varphi_1 : M \longrightarrow N$ is an R-module homomorphism such that $\varphi_1(am) = \varphi_0(a)\, \varphi_1(m)$ and the following diagram commutes :

$$\begin{array}{ccc} A & \xrightarrow{\varphi_0} & B \\ d \downarrow & & \downarrow \delta \\ M & \xrightarrow{\varphi_1} & N \end{array}$$

When $\varphi_0 = f$, φ will also be referred to as f-derivation module homomorphism. If the derivation module homomorphism $\varphi : (A, M, d) \longrightarrow (A, N, \delta)$ is such that $\varphi_0 = I_A$ then we have $\varphi_1 d = \delta$ and φ will be referred to as an A - derivation module homomorphism.

Remark (5) : The class of all R-derivation modules and R-derivation module homomorphisms forms a category and we shall denote it by $\mathcal{D}$.

Remark (6) : The class of all A-derivation modules and A-derivation module homomorphisms forms a category and we shall denote it by $\mathcal{D}(A)$.

Remarks (7) : Let $\{(A, M_\alpha, d_\alpha)\}_{\alpha \in I}$ be a family of A-derivation modules. Then $(A, \prod_\alpha M_\alpha, (d_\alpha)_\alpha)$ is a derivation module of A and is the 'product' of the family in $\mathcal{D}(A)$.

Remarks (8) : Let (B, N, δ) be a B-derivation module and $f : A \longrightarrow B$ be an R-algebra homomorphism. Then N can be considered as an A-module via $f : A \longrightarrow B$ by defining the scaler multiplication as $a.n = f(a)n$ for $a \in A$ and $n \in N$. Moreover, $\delta.f : A \longrightarrow N$ is an R-derivation making $(A, N, \delta f)$ an A-derivation module.

Remark (9) : Let (A, M, d) be an R-derivation module. Defining $m.m' = o$ and $d_1(m) = o$ for all $m, m' \in M$, an R-derivation module (A, M, d) offers an R-complex (X, δ) where $X_o = A$, $X_1 = M$, $X_n = 0$ for $n \geqslant 2$ and $\delta_o = d$, $\delta_n = 0$ for $n \geqslant 1$. Then an f-derivation module homomorphism $\varphi : (A, M, d) \longrightarrow (B, N, \partial)$ is the f-complex homomorphism $\varphi : (X, \delta) \rightarrow (Y, \Delta)$ where $\varphi_o = f$, $\varphi_1 = \varphi_1$, $\varphi_n = 0$ for $n \geqslant 2$ and $Y_o = B$, $Y_1 = N$, $Y_n = 0$ for $n \geqslant 2$ and $\Delta_o = \partial$, $\Delta_n = 0$ for $n \geqslant 1$.

Definition (2.5) : Let (A, M, d) and (B, N, δ) be R-derivation modules. Let φ : (A, M, d) $\longrightarrow$ (B, N, δ) be the derivation module homomorphism. Then the derivation module (B, N, δ) is said to be φ - simple if and only if N is generated by $\delta(B) \cup \varphi_1(M)$ as a B-module. We shall usually denote φ_o and φ_1 by the same symbol φ .

Proposition (2.1) : Let (A, M, d) be an R-derivation module. Then for any R-derivation module (B, N, δ) and derivation module homomorphism φ : (A, M, d) $\longrightarrow$ (B, N, δ), there exists a φ^* - simple derivation module (B, N^*, δ^*) and a B-derivation module monomorphism j : (B, N^*, δ^*) $\longrightarrow$ (B, N, δ) such that j $\varphi^* = \varphi$.

Proof : Denote by N^* the B - submodule of N generated by $\delta(B) \cup \varphi(M)$. Since N^* is a B - submodule of N and since $\delta(B) \subseteq N^*$ we have that (B, N^*, δ^*) is an R-derivation module where $\delta^* : B \longrightarrow N^*$ is defined as $\delta^* = \delta$.

Define φ^* : (A, M, d) $\longrightarrow$ (B, N^*, δ^*) as $\varphi^* = \varphi$. Then clearly (B, N^*, δ^*) is φ^*-simple. Let j : $N^* \longrightarrow$ N denote the natural inclusion. Then j : . . is a B-derivation module monomorphism satisfying j $\varphi^* = \varphi$; i.e. making the following diagram commutative.

$$\begin{array}{ccc} (A, M, d) & \xrightarrow{\varphi^*} & (B, N^*, \delta^*) \\ & \searrow^{\varphi} & \downarrow j \\ & & (B, N, \delta) \end{array}$$

Hence, the proof.

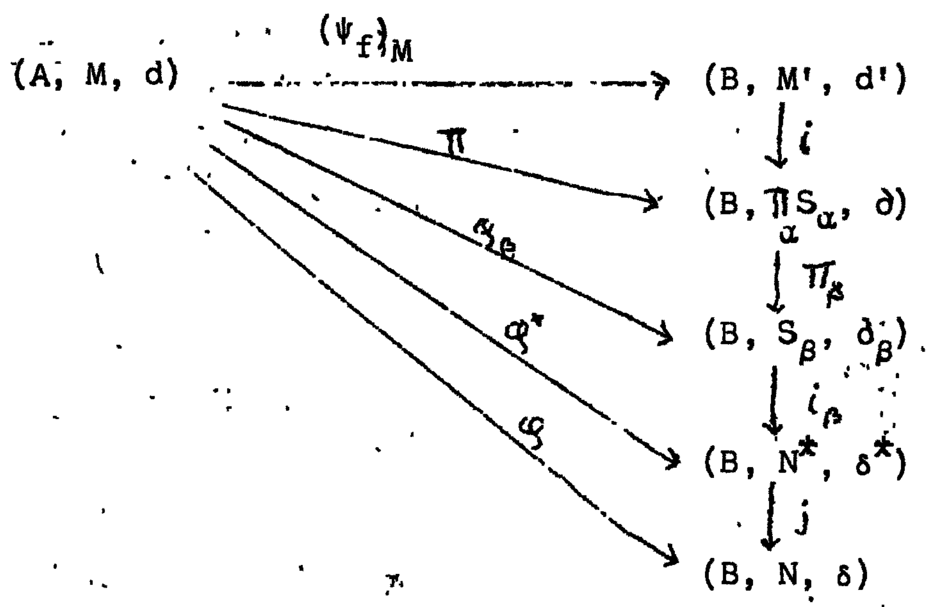
Now we are going to prove that any R-algebra homomorphism $f : A \to B$ in $\mathcal{A}$ gives rise to a natural covariant functor from the category of A - derivation modules to the category of B - derivation modules.

<u>Proposition (2.2)</u> : Let $f : A \longrightarrow B$ be an R-algebra homomorphism. For any R-derivation module (A, M, d), there exists an R-derivation module (B, M', d') and an f-derivation module homomorphism $(\psi_f)_M : (A, M, d) \longrightarrow (B, M', d')$ in $\mathcal{D}$ such that for any R-derivation module (B, N, δ) and an f-derivation module homomorphism $\varphi : (A, M, d) \longrightarrow (B, N, \delta)$, there exists a unique B - derivation module homomorphism $\varphi'' : (B, M', d') \longrightarrow (B, N, \delta)$ satisfying $\varphi'' \ (\psi_f)_M = \varphi$. Moreover, (B, M', d') and $(\psi_f)_M$ are unique in the sense that if there exists another such B-derivation module $(B, \overline{M}, \overline{d})$ and an f-derivation module homomorphism $h_M : (A, M, d) \longrightarrow (B, \overline{M}, \overline{d})$ then there exists a B-derivation module isomorphism $i : (B, M', d') \longrightarrow (B, \overline{M}, \overline{d})$ satisfying $i.(\psi_f)_M = h_M$.

Proof : If $\varphi : (A, M, d) \longrightarrow (B, S, \partial)$ is any f-derivation module homomorphism and if (B, S, ∂) is φ - simple then $|S| \leq |B| \aleph_0$ holds. So there exists a family $\{(B, S_\alpha, \partial_\alpha)\}_{\alpha \in I}$ of φ_α - simple derivation modules indexed by the set I such that for any φ - simple B-derivation module (B, S, ∂), there exists a B-derivation module isomorphism $i_\alpha : (B, S_\alpha, \partial_\alpha) \longrightarrow (B, S, \partial)$ for some $\alpha \in I$ such that $i_\alpha \cdot \varphi_\alpha = \varphi$.

I is nonempty, because the trivial B-derivation module (B, O, o) is φ - simple where $\partial = o$ and $\varphi : (A,M,d) \longrightarrow (B,O,o)$ is f - derivation module defined by $\varphi = (f,o)$.

Now consider the derivation $\partial : B \longrightarrow \prod_\alpha S_\alpha$ defined as $\partial(b) = (\partial_\alpha(b))_\alpha$. This gives the product $(B, \prod_\alpha S_\alpha, \partial)$ of the representative family $\{(B, S_\alpha, \partial_\alpha)\}_{\alpha \in I}$ of φ_α - simple B-derivation modules. Let $\pi : (A, M, d) \rightarrow (B, \prod_\alpha S_\alpha, \partial)$ be defined as $\pi(a) = f(a)$ and $\pi(m) = (\varphi_\alpha(m))_\alpha$ for $a \in A$ and $m \in M$. Let M' denote the B-submodule of $\prod_\alpha S_\alpha$ generated by $\delta(B) \cup \pi(M)$. Since $\delta(B) \subseteq M'$, (B, M', d') is a derivation module where $d' : B \longrightarrow M'$ is defined as $d' = \partial$. Define $(\psi_f)_M : (A, M, d) \longrightarrow (B, M', d')$ as $(\psi_f)_M = \pi$. Then $(\psi_f)_M$ is an f-derivation module homomorphism and (B, M', d') is $(\psi_f)_M$ - simple.



Now, for any derivation module (B, N, δ), any f-derivation module homomorphism $\varphi : (A, M, d) \longrightarrow (B, N, \delta)$, there exists by prop(2.1) a φ^* - simple derivation module (B, N^*, δ^*) and B - derivation module monomorphism

$j : (B, N^*, \delta^*) \longrightarrow (B, N, \delta)$ such that $j\,\varphi^* = \varphi$.

Since the derivation module (B, N^*, δ^*) is φ^* - simple, there exists some φ_β - simple derivation module $(B, S_\beta, \partial_\beta)$ in the representative family $\{(B, S_\alpha, \partial_\alpha)\}_{\alpha \in I}$ for some $\beta \in I$ and B-derivation module isomorphism

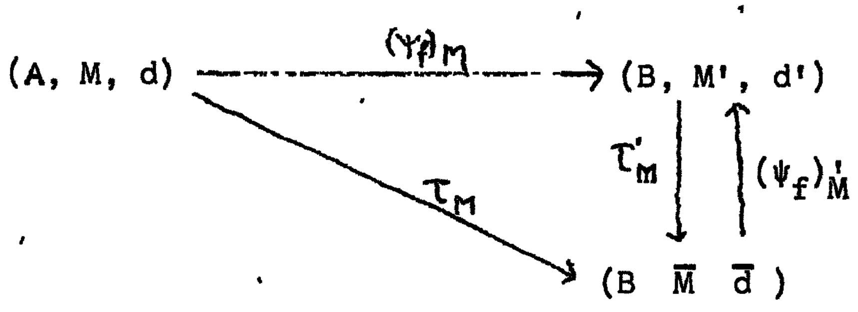
$i_\beta : (B, S_\beta, \partial_\beta) \longrightarrow (B, N^*, \delta^*)$ with $i_\beta . \varphi_\beta = \varphi^*$.

Let $\pi_\beta : (B, \prod_\alpha S_\alpha, \partial) \longrightarrow (B, S_\beta, \partial_\beta)$ be identity on B and β^{th} projection on $\prod_\alpha S_\alpha$. Then $\pi_\beta\, \pi = \varphi_\beta$.
Let $i : (B, M', d') \longrightarrow (B, \prod_\alpha S_\alpha, \partial)$ denote the inclusion mapping. Then $i\,(\psi_f)_M = \pi$.

Let us put $\varphi'' = j \;\; i_\beta \;\; \pi_\beta \; i$. Then, since all the small triangles in the above diagram commute, the outermost triangle also commutes. Therefore, $\therefore \; \varphi''.(\psi_f)_M = \varphi$ i.e. there exists a B-derivation module homomorphism $\varphi'' : (B, M', d') \longrightarrow (B, N, \delta)$ such that $\varphi''.(\psi_f)_M = \varphi$.

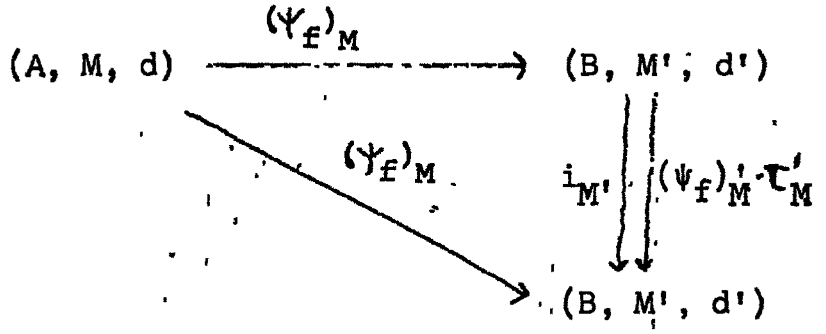
The uniqueness of φ'' follows from the fact that (B, M', d') is $(\psi_f)_M$ - simple from the definition of (B,M',d').

Finally to prove the uniqueness of (B, M', d') and $(\psi_f)_M$, let $(B, \overline{M}, \overline{d})$ and an f-derivation module homomorphism $\tau_M : (A, M, d) \longrightarrow (B, \overline{M}, \overline{d})$ be another such, then there exists a unique derivation module homomorphism $\tau'_M : (B, M', d') \longrightarrow (B,\overline{M},\overline{d})$ such that $\tau'_M \, (\psi_f)_M = \tau_M$ and there also exists a derivation module homomoraphism $(\psi_f)'_M : (B, \overline{M}, \overline{d}) \longrightarrow (B, M', d')$ such that $(\psi_f)'_M \, \tau_M = (\psi_f)_M$ as in the following diagram :



Now $(\psi_f)'_M \cdot \tau'_M : (B, M', d') \longrightarrow (B, M', d')$ is a B - derivation module homomorphism satisfying

$(\psi_f)'_M \;\; \tau'_M \;\; (\psi_f)_M \; = (\psi_f)_M$



But the identity mapping $i_{M'}$ on (B, M', d') is also another such derivation module homomorphism. Hence, by uniqueness of such B-derivation module homomorphisms we have $(\psi_f)'_M = i_{M'}$

In the same way $\tau'_M\ (\psi_f)'_M = i_{\overline{M}}$.
Hence $\tau'_M : (B, M', d') \longrightarrow (B, \overline{M}, \overline{d})$ is a derivation module isomorphism such that $\tau'_M\ (\psi_f)_M = \tau_M$. This completes the proof.

<u>Proposition (2.3)</u> : Let (A, M, d) and (A, N, δ) be derivation modules and (B, M', d') and (B, N', δ') be the corresponding derivation modules. Then for any A-derivation module homomorphism $\lambda : (A, M, d) \longrightarrow (A, N, \delta)$, there exists a unique B-derivation module homomorphism $\lambda' : (B, M', d') \longrightarrow (B, N', \delta')$ such that $\lambda'\ (\psi_f)_M = (\psi_f)_N\ \lambda$.

<u>Proof</u> : The composition $(\psi_f)_N\ \lambda : (A, M, d) \longrightarrow (B, N', \delta')$ is an f-derivation module homomorphism. Therefore, there exists by Prop (2.2) a unique B-derivation module homomorphism $\lambda' : (B, M', d') \longrightarrow (B, N', \delta')$ such that the following diagram commutes :

$$\begin{array}{ccc}
(A,\ M,\ d) & \xrightarrow{(\psi_f)_M} & (B,\ M',\ d') \\
\Big\downarrow \lambda & & \Big\downarrow \lambda' \\
(A,\ N,\ \delta) & \xrightarrow{(\psi_f)_N} & (B,\ N',\ \delta')
\end{array}$$

Hence, the proof.

Define $f_* : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ as $f_*((A,M,d)) = (B,\ M',\ d')$ [as defined in Prop (2.2)] and $f_*(\lambda) = \lambda'$ [as defined in Prop(2.3)], for all $(A,\ M,\ d) \in \mathcal{D}(A)$ and for all $\lambda \in \mathcal{D}(A)$.

If $I : (A,\ M,\ d) \longrightarrow (A,\ M,\ d)$ is the identity in $\mathcal{D}(A)$, then $f_*(I) = I' : (B,\ M',\ d') \longrightarrow (B,\ M',\ d')$ is also the identity in $\mathcal{D}(B)$.

Let $(A,\ M,\ d)$, $(A,\ N,\ \delta)$ and $(A,\ L,\ \partial)$ be derivation modules in $\mathcal{D}(A)$ and let $\varphi : (A,\ M,\ d) \longrightarrow (A,\ N,\ \delta)$ and $\psi : (A,\ N,\ \delta) \longrightarrow (A,\ L,\ \partial)$ be morphisms in $\mathcal{D}(A)$. Then

$$\begin{array}{ccccc}
(A,\ M,\ d) & \xrightarrow{(\psi_f)_M} & (B,\ M',\ d') & & \\
\Big\downarrow \varphi & & \Big\downarrow \varphi' & & \\
(A,\ N,\ \delta) & \xrightarrow{(\psi_f)_N} & (B,\ N',\ \delta') & & (\psi\ \varphi)' \\
\Big\downarrow \psi & & \Big\downarrow \psi' & & \\
(A,\ L,\ \partial) & \xrightarrow{(\psi_f)_L} & (B,\ L',\ \partial') & &
\end{array}$$

We have $(\psi_f)_L\ \psi\ \varphi = \psi'\ (\psi_f)_N\ \varphi = \psi'\ \varphi'\ (\psi_f)_M.$

Similarly we have $(\psi_f)_L\, \psi\, \varphi = (\psi\, \varphi)'\, (\psi_f)_M$. By the uniqueness of such morphisms we have $(\psi\, \varphi)' = \psi'\, \varphi'$.

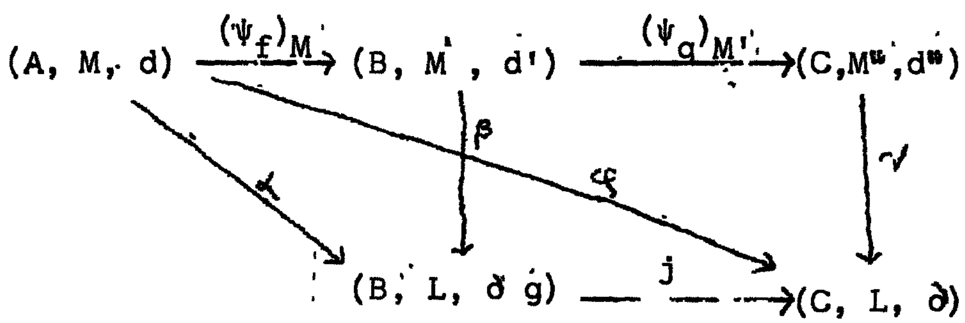
$$\text{i.e. } f_*(\psi\, \varphi) = f_*(\psi)\ \ f_*(\varphi).$$

Thus we have proved :

<u>Theorem (2.1)</u> : If $f : A \longrightarrow B$ is an algebra homomorphism, then there exists a covariant functor $f_* : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ defined by $f_*((A,M,d)) = (B, M', d')$ and $f_*(\lambda) = \lambda'$ for all $(A,M,d) \in \mathcal{D}(A)$ and $\lambda \in \mathcal{D}(A)$.

<u>Proposition (2.4)</u> : If A, B, C are unitary commutative R-algebras, $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are unitary algebra homomorphisms, then there exists a natural equivalence $c_{fg} : (g\, f)_* \longrightarrow g_*\, f_*$.

<u>Proof</u> : Let (A,M,d) and (C,L,∂) be R-derivation modules. Let $\varphi : (A, M, d) \longrightarrow (C, L, \partial)$ be a $g\,f$ - derivation module homomorphism. The C-module L can be considered as B-module via $g : B \longrightarrow C$ and $\partial\, g : B \longrightarrow L$ is R-derivation and $(B,L,\partial\, g)$ is a B-derivation module.



Define $\alpha : (A,M,d) \longrightarrow (B,L,\partial g)$ as $\alpha = (f, \varphi_1)$. Since α is f-derivation module homomorphism, then by Prop (2.2) there exists a unique B-derivatión module homomorphism $\beta : (B, M', d') \longrightarrow (B, L, \partial.g)$ such that $\beta\, (\psi_f)_M = \alpha$. Now define $j : (B, L, \partial g) \longrightarrow (C, L, \partial)$ as $j = (g, I)$. Then $j\,\beta : (B, M', d') \longrightarrow (C, L, \partial)$ is a g-derivation module homomorphism. Therefore again by Prop (2.2) there exists a unique C - derivation module homomorphism $\gamma : (C,M'',d'') \longrightarrow (C,L,\partial)$ such that $\gamma\, (\psi_g)_{M'} = j.\beta$. Moreover $j\,\alpha = \varphi$ because $j.\alpha\,(a\,m) = j(f(a).\,\varphi_1(m)) = g.f(a)\;\varphi_1(m) = \varphi(am)$ for $a \in A$ and $m \in M$.

We claim that $\gamma\, (\psi_g)_{M'}.(\psi_f)_M = \varphi$.
In fact $\gamma\, (\psi_g)_{M'}.(\psi_f)_M = j\,\beta\,(\psi_f)_M = j\,\alpha = \varphi$.
The uniqueness of γ follows from the fact that (C,M'', d'') is $(\psi_g)_{M'}\,(\psi_f)_M$ - simple.

On the other hand, let $(\psi_{gf})_M : (A,M,d) \longrightarrow (C,\overline{M},\overline{d})$ be the g.f-derivation module homomorphism where $(C,\overline{M},\overline{d}) = (gf)_*(A,M,d)$. For the g.f - derivation module homomorphism $\varphi : (A,M,d) \longrightarrow (C,L,\partial)$ there exists by Prop (2.2) a unique derivatiòn module homomorphism $\tau : (C,\overline{M},\overline{d}) \longrightarrow (C,L,\partial)$ such that $\tau\,(\psi_{gf})_M = \varphi$. i.e. making the following diagram commutative.

$$
\begin{array}{ccc}
(A,\ M,\ d) & \xrightarrow{(\psi_{gf})_M} & (C,\ \overline{M},\ \overline{d}) \\
 & \searrow^{\varphi} & \downarrow \tau \\
 & & (C,\ L,\ \partial)
\end{array}
$$

By the uniqueness of such C-derivation modules and C-derivation module homomorphisms, there exists a C-derivation module isomorphism

$$(c_{f,g})_M : (C, \bar{M}, \bar{d}) \longrightarrow (C, M'', d'') \text{ such that}$$

$(c_{fg})_M (\psi_{gf})_M = (\psi_g)_{M'} (\psi_f)_M$ i.e. making the following diagram commutative :

$$\begin{array}{ccl} (A,M,d) & \xrightarrow{(\psi_{gf})_M} & (C,\bar{M},\bar{d}) = (gf)_*(A,M,d) \\ & \searrow^{(\psi_g)_{M'}\cdot(\psi_f)_M} & \Big\downarrow (c_{fg})_M \\ & & (C, M'', d'') = g_* f_* (A,M,d) \end{array}$$

This means that

$$(c_{fg})_M : (gf)_* (A,M,d) \longrightarrow g_* f_* (A,M,d) \text{ is an isomorphism.}$$

Thus, the natural transformation $c_{fg} : (gf)_* \longrightarrow g_* f_*$ is the natural equivalence.

Let $\mathcal{A}$ denote the category of all unitary commutative algebras over R. Let $\mathcal{D}$ denote the category of all R-derivation modules. Consider the functor $P : \mathcal{D} \to \mathcal{A}$ defined as $P((A,M,d)) = A$ and $P(\varphi) = \varphi_0$ where $\varphi : (A, M, d) \longrightarrow (B, N, \delta)$ is a derivation module homomorphism. Then the fibre $P^{-1}(A)$ is the category $\mathcal{D}(A)$ of A-derivation modules and A-derivation module homomorphisms. Let $J_A : \mathcal{D}(A) \to \mathcal{D}$ denote the inclusion functor. Our claim is :

<u>Theorem (2.2)</u> : The functor $P : \mathcal{D} \to \mathcal{A}$ admits an opcleavage $\{f_*, \psi_f, c_{fg}\}$.

<u>Proof</u> : For each $f : A \longrightarrow B$ in $\mathcal{A}$ and for any $(A,M,d) \in \mathcal{D}(A)$ there exists a unique $f_*(A,M,d) = (B,M',d') \in \mathcal{D}(B)$ and an f-derivation module homomorphism $(\psi_f)_M : (A,M,d) \longrightarrow f_*(A,M,d)$ in $\mathcal{D}$ such that $P((\psi_f)_M) = f$ by Prop (2.2). For any morphism $\lambda : (A,M,d) \longrightarrow (A,N,\delta)$ in $\mathcal{D}(A)$ these exists a unique morphism $\lambda' = f_*(\lambda) : f_*(A, M, d) \longrightarrow f_*(A, N, \delta)$ in $\mathcal{D}(B)$ by Prop(2.3). Thus each morphism $f : A \longrightarrow B$ in $\mathcal{A}$ gives rise to a functor $f_* : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$. There exists a natural transformation $\psi_f : J_A \longrightarrow J_B f_*$ satisfying the condition that $P((\psi_f)_M) = f$ for all $(A,M,d) \in \mathcal{D}(A)$ by Prop (2.2).

For any f-derivation module homomorphism $\varphi : (A,M,d) \longrightarrow (B,N,\delta)$ satisfying $P(\varphi) = f$, there exists a unique B-derivation module homomorphism $\varphi'' : f_*(A,M,d) \longrightarrow (B,N,\delta)$ in $\mathcal{D}(B)$ such that $\varphi''.(\psi_f)_M = \varphi$ by Prop (2.2), i.e. making the following diagram commutative.

$$\begin{array}{ccc} (A,M,d) & \xrightarrow{(\psi_f)_M} & f_*(A,M,d) \\ & \searrow^{\varphi} & \downarrow \varphi'' \\ & & (B,N,\delta) \end{array}$$

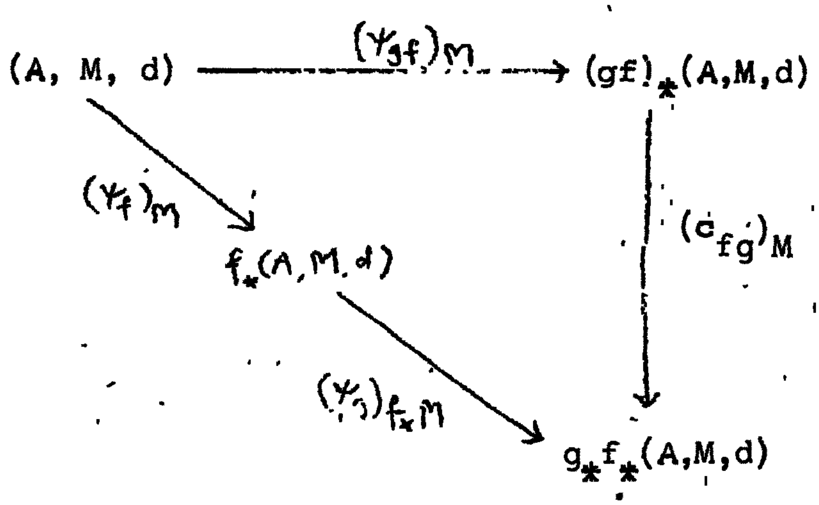
Now consider the composition $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. Then for each (A,M,d) in $\mathcal{D}(A)$ there is a uniquely determined morphism

$$(c_{fg})_M : (g.f)_*(A,M,d) \longrightarrow g_* f_*(A,M,d) \text{ in } \mathcal{D}(C)$$

such that

$(c_{fg})_M \; (\psi_{gf})_M = (\psi_g)_{f_* M} \; (\psi_f)_M$ by Prop (2.4).

i.e. the following diagram commutes :



It can be easily seen that $(c_{fg})_M$ are the components of a natural transformation $c_{fg} : (gf)_* \longrightarrow g_* f_*$.
This natural transformation c_{fg} is a natural equivalence by Prop (2.4). This proves that the functor $P : \mathcal{D} \to \mathcal{A}$ admits an opcleavage $\{ f_*, \psi_f, c_{fg} \}$.

In the following, we shall prove that any unitary algebra homomorphism $f:A \longrightarrow B$ in $\mathcal{A}$ gives rise to a covariant functor f^* from the category of B-derivation modules to the category of A - derivation modules.

<u>Proposition (2.5)</u> : Let $f:A \longrightarrow B$ be a unitary algebra homomorphism. Then for any derivation module (B,N,δ) in $\mathcal{D}(B)$, there exists a derivation module $(A,\bar{N},\bar{\delta})$ in $\mathcal{D}(A)$ and f-derivation module homomorphism $(\theta_f)_N : (A, \bar{N}, \bar{\delta}) \longrightarrow (B, N, \delta)$.

<u>Proof</u> : Let (B, N, δ) be a derivation module in $\mathcal{D}(B)$. Consider the derivation module $(A, \overline{N}, \overline{\delta})$ where $\overline{N} = N$ as an A-module and $\overline{\delta} = \delta f$. Then $(A, \overline{N}, \overline{\delta})$ is in $\mathcal{D}(A)$. Define the mapping $(\theta_f)_N : (A, \overline{N}, \overline{\delta}) \longrightarrow (B, N, \delta)$ as $(\theta_f)_N = (f, I)$. Clearly $(\theta_f)_N$ is f-derivation module homomorphism because the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \delta f = \overline{\delta}}\downarrow & & \downarrow{\scriptstyle \delta} \\ \overline{N} & \underline{I} & N \end{array}$$

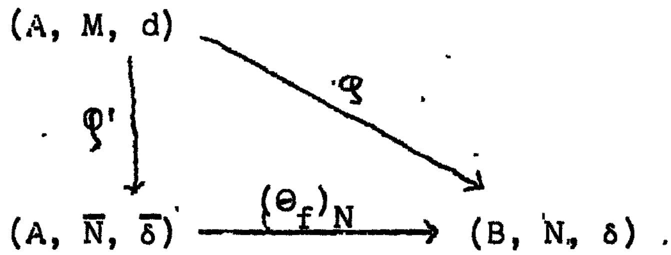
Thus for any derivation module (B, N, δ) in $\mathcal{D}(B)$ there exists a derivation module $(A, \overline{N}, \overline{\delta})$ in $\mathcal{D}(A)$ together with an f-derivation module homomorphism $(\theta_f)_N : (A, \overline{N}, \overline{\delta}) \longrightarrow (B, N, \delta)$.

<u>Proposition (2.6)</u> : Let $f: A \longrightarrow B$ be an algebra homomorphism and (A, M, d) and (B, N, δ) be derivation modules. Then for any f-derivation module homomorphism $\varphi : (A, M, d) \longrightarrow (B, N, \delta)$ there exists a unique A-derivation module homomorphism $\varphi' = (A, M, d) \longrightarrow (A, \overline{N}, \overline{\delta})$ such that $(\theta_f)_N \, \varphi' = \varphi$.

<u>Proof</u> : Let $\varphi : (A, M, d) \longrightarrow (B, N, \delta)$ be f-derivation module homomorphism. Then there exists by Prop (2.5) a derivation module $(A, \overline{N}, \overline{\delta})$ together with an f-derivation module homomorphism $(\theta_f)_N : (A, \overline{N}, \overline{\delta}) \longrightarrow (B, N, \delta)$.

Define $\varphi' : (A, M, d) \longrightarrow (A, \bar{N}, \bar{\delta})$ as $\varphi' = (I, \varphi)$. Since $\varphi\, d = \delta\, f$ holds we have that $\varphi' : (A, M, d) \longrightarrow (A, \bar{N}, \bar{\delta})$ is an A - derivation module homomorphism.

Now we claim that $(\Theta_f)_N\, \varphi' = \varphi$. Let $a.m \in M$ be any element. Then $(\Theta_f)_N\, \varphi'\, (a.m) = (\Theta_f)_N\, (a.\varphi(m)) = f(a).\, \varphi(m) = \varphi(a.m)$ for $a \in A$ and $m \in M$. Thus the following diagram commutes :



To prove the uniqueness of φ' which makes the above diagram commutative, suppose there is another such A-derivation module homomorphism $\varphi'' : (A, M, d) \longrightarrow (A, \bar{N}, \bar{\delta})$ satisfying $(\Theta_f)_N\, \varphi'' = \varphi$.

$\varphi''_o = \varphi'_o = I_A$. For $m \in M$ We have $(\Theta_f)_N\, \varphi''(m) = (\Theta_f)_N\, \varphi'(m)$ But, since $(\Theta_f)_N \mid \bar{N}$ = identity we have $\varphi''(m) = \varphi'(m)$. Thus $\varphi''_1 = \varphi'_1$. Thus $\varphi'' = \varphi'$. Hence, such φ' is unique.

<u>Proposition (2.7)</u> : Let (B, M, d) and (B, N, δ) be derivation modules in $\mathcal{D}(B)$ and let $(A, \bar{M}, \bar{d})$ and $(A, \bar{N}, \bar{\delta})$ be the corresponding derivation modules in $\mathcal{D}(A)$. Then for any B-derivation module homomorphism $k : (B,M,d) \longrightarrow (B,N,\delta)$ there

exists a unique A-derivation module homomorphism $\bar{k} : (A, \bar{M}, \bar{d}) \longrightarrow (A, \bar{N}, \bar{\delta})$ such that $(\theta_f)_N \; \bar{k} = k \; (\theta_f)_M$.

<u>Proof</u> : The composition $k \; (\theta_f)_M : (A, \bar{M}, \bar{d}) \longrightarrow (B, N, \delta)$ is an f-derivation module homomorphism. Therefore, there exists by Prop (2.6) a unique A-derivation module homomorphism $\bar{k} : (A, \bar{M}, \bar{d}) \longrightarrow (A, \bar{N}, \bar{\delta})$ such that the following diagram commutes :

$$\begin{array}{ccc} (A, \bar{M}, \bar{d}) & \xrightarrow{(\theta_f)_M} & (B, M, d) \\ \bar{k} \downarrow & & \downarrow k \\ (A, \bar{N}, \bar{\delta}) & \xrightarrow{(\theta_f)_N} & (B, N, \delta) \end{array}$$

Hence the proof.

Define $f^* : \mathcal{D}(B) \to \mathcal{D}(A)$ as $f^*((B,M,d)) = (A, \bar{M}, \bar{d})$ [as defined in Prop (2.5)] and $f^*(k) = \bar{k}$ [as defined in Prop (2.7)] for all $(B, M, d) \in \mathcal{D}(B)$ and for all $k \in \mathcal{D}(B)$.

If $I : (B, M, d) \longrightarrow (B, M, d)$ is the identity in $\mathcal{D}(B)$, the $f^*(I) = \bar{I} : (A,\bar{M},\bar{d}) \longrightarrow (A, \bar{M}, \bar{d})$ is also the identity in $\mathcal{D}(A)$.

Let (B, M, d), (B, N, δ) and (B, L, ∂) be derivation modules in $\mathcal{D}(B)$ and let $\varphi : (B, M, d) \longrightarrow (B, N, \delta)$ and $\psi : (B, N, \delta) \longrightarrow (B, L, \partial)$ be morphisms in $\mathcal{D}(B)$. Then

$$\begin{array}{ccc}
(A,\overline{M},\overline{d}) & \xrightarrow{(\theta_f)_M} & (B,\ M,\ d) \\
\overline{\varphi}\downarrow & & \downarrow\varphi \\
(A,\ \overline{N},\ \overline{\delta}) & \xrightarrow{(\theta_f)_N} & (B,\ N,\ \delta) \\
\overline{\psi}\downarrow & & \downarrow\psi \\
(A,\ \overline{L},\ \overline{\partial}) & \xrightarrow{(\theta_f)_L} & (B,\ L,\ \partial)
\end{array}$$

We have $\psi\ \varphi(\theta_f)_M = \psi(\theta_f)_N\ \overline{\varphi} = (\theta_f)_L\ \overline{\psi}\ \overline{\varphi}$. Similarly we have $\psi.\varphi\ (\theta_f)_M = (\theta_f)_L\ (\overline{\psi.\varphi})$.

By the uniqueness of such morphisms we have

$$\overline{(\psi\ \varphi)} = \overline{\psi}\ \overline{\varphi}$$

$$\text{i.e.}\quad f^*(\psi\ \varphi) = f^*(\psi)\ f^*(\varphi).$$

Thus we have proved :

<u>Theorem (2.3)</u> : If $f : A \longrightarrow B$ is an algebra homomorphism then there exists a covariant functor $f^* : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ defined by $f^*(B,\ M,\ d) = (A,\ \overline{M},\ \overline{d})$ and $f^*(k) = \overline{k}$ for all $(B,\ M,\ d) \in \mathcal{D}(B)$ and $k \in \mathcal{D}(B)$.

<u>Proposition (2.8)</u> : If A, B, C are unitary commutative R-algebras and $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be unitary algebra homomorphisms. Then $f^*\ g^* = (g\ f)^*$.

<u>Proof</u>: For this take a derivation module (C, M, d) in $\mathcal{D}(C)$. Then $g^* : \mathcal{D}(C) \longrightarrow \mathcal{D}(B)$ associates with (C, M, d) the derivation module $(B, \overline{M}, \overline{d})$ in $\mathcal{D}(B)$. Again $f^* : \mathcal{D}(B) \longrightarrow \mathcal{D}(A)$ associates with $(B, \overline{M}, \overline{d})$ the derivation module $(A, \bar{\bar{M}}, \bar{\bar{d}})$ in $\mathcal{D}(A)$. Similarly $(g.f)^* : \mathcal{D}(C) \longrightarrow \mathcal{D}(A)$ associates with (C, M, d) the derivation module $(A, \widetilde{M}, \widetilde{d})$ in $\mathcal{D}(A)$.

Define the mapping

$$(d_{fg})_M : (A, \bar{\bar{M}}, \bar{\bar{d}}) \longrightarrow (A, \widetilde{M}, \widetilde{\delta})$$

as $(d_{fg})_M \mid A$ = identity and $(d_{fg})_M \mid M$ = identity.

$$f^* g^* (C,M,d) = (A, \bar{\bar{M}}, \bar{\bar{d}}) \xrightarrow{(\theta_f)_{\overline{M}}} (B, \overline{M}, \overline{d}) \xrightarrow{(\theta_g)_M} (C,M,d)$$

$$(d_{fg})_M \downarrow$$

$$(g\ f)^*(C,M,d) = (A, \widetilde{M}, \widetilde{d}) \xrightarrow{(\theta_{gf})_M} (C,M,d)$$

Obviously $(d_{fg})_M$ is the identity A-derivation module isomorphism in $\mathcal{D}(A)$ on the derivation module $(A, \bar{\bar{M}}, \bar{\bar{d}}) = (A, \widetilde{M}, \widetilde{d})$.

Again, for every C - derivation module homomorphism

$$\varphi : (C, M, d) \longrightarrow (C, N, \delta)$$

in $\mathcal{D}(C)$, the following diagram is commutative. :

$$\begin{array}{ccc} (A,\bar{\bar{M}},\bar{\bar{d}}) & \xrightarrow{(d_{fg})_M} & (A,\tilde{M},\tilde{d}) \\ \bar{\bar{\varphi}} \downarrow & & \downarrow \tilde{\varphi} \\ (A,\bar{\bar{N}},\bar{\bar{\delta}}) & \xrightarrow{(d_{fg})_N} & (A,\tilde{N},\tilde{\delta}) \end{array}$$

For let $a.m \in \bar{\bar{M}} = M$ be any element where $a \in A$ and $m \in M$. Then $(d_{fg})_N . \bar{\bar{\varphi}}\,(a.m) = (d_{fg})_N\,(a.\varphi(m)) = \tilde{\varphi}\,(a.m) = \tilde{\varphi}\,(d_{fg})_M(a.m)$. Thus $(d_{fg})_N\ \bar{\bar{\varphi}} = \tilde{\varphi}\ (d_{fg})_M$ i.e. the above diagram commutes. Thus $d_{fg} : f^* g^* \longrightarrow (g\ f)^*$ is the identity natural equivalence. Hence, $f^* g^* = (g\ f)^*$.

<u>Theorem (2.4)</u> : The functor $P : \mathcal{D} \longrightarrow \mathcal{A}$ admits a cleavage $\{f^*, \theta_f, d_{fg}\}$.

<u>Proof</u> : For each $f : A \to B$ in $\mathcal{A}$ and for any (B, M, d) in $\mathcal{D}(B)$ there exists a unique $f^*(B, M, d) = (A, \bar{M}, \bar{d}) \in \mathcal{D}(A)$ and an f - derivation module homomorphism $(\theta_f)_M : f^*(B,M,d) \to (B,M,d)$ in $\mathcal{D}$ such that $P((\theta_f)_M) = f$ by Prop (2.5).

For any $k = (B, M, d) \longrightarrow (B, N, \delta)$ in $\mathcal{D}(B)$ there exists a unique morphism :-

$\bar{k} = f^*(k)$: $f^*(k)$: $f^*(B,M,d) \longrightarrow f^*(B,N,\delta)$ in $\mathcal{D}(A)$ by Prop (2.7). Thus each morphism $f : A \longrightarrow B$ in $\mathcal{A}$ gives rise to a functor $f^* : \mathcal{D}(B) \rightarrow \mathcal{D}(A)$. There exists a natural transformation $\Theta_f : J_A f^* \longrightarrow J_B$ satisfying the condition that $P((\Theta_f)_N) = f$ for all $(B, N, \delta) \in \mathcal{D}(B)$ by Prop (2.5).

For any f-derivation module homomorphism $\varphi : (A,M,d) \longrightarrow (B, N, \delta)$ satisfying $P(\varphi) = f$, there exists a unique A-derivation module homomorphism $\varphi' : (A,M,d) \rightarrow f^*(B,N,\delta)$ in $\mathcal{D}(A)$ such that $(\Theta_f)_N \varphi' = \varphi$ by Prop (2.6). i.e. making the following diagram commutative.

$$\begin{array}{ccc} (A, M, d) & & \\ \downarrow \varphi' & \searrow \varphi & \\ f^*(B,N,\delta) & \xrightarrow{(\Theta_f)_N} & (B,N,\delta) \end{array}$$

Now consider the composition $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. Then for each (C,M,d) in $\mathcal{D}(C)$ there is a uniquely determined morphism $(d_{fg})_M : f^*.g^*(C,M,d) \longrightarrow (g f)^*(C,M,d)$ in $\mathcal{D}(A)$ such that

$$(\Theta_{gf})_M \quad (d_{fg})_M = (\Theta_g)_M \quad (\Theta_f)_{g^* M}$$

by prop (2.8); i.e. the following diagram commutes.

$$\begin{array}{ccccc}
f^* g^*(C,M,d) & \xrightarrow{(\theta_f)_{g^*M}} & g^*(C,M,d) & & \\
\downarrow (d_{fg})_M & & & \searrow (\theta_g)_M & \\
(g\, f)^*(C,M,d) & & \xrightarrow{(\theta_{gf})_M} & & (C,M,d)
\end{array}$$

It can be easily seen that $(d_{fg})_M$ are the components of a natural transformation $d_{fg} : f^* g^* \longrightarrow (g\, f)^*$. This natural transformation d_{fg} is the identity natural equivalence by Prop (2.8). This proves that the functor $P : \mathcal{D} \to \mathcal{A}$ admits a split cleavage $\{f^*, \theta_f, d_{fg}\}$.

If $i_A : A \longrightarrow A$ is the identity morphism in $\mathcal{A}$, then $(i_A)^* : \mathcal{D}(A) \longrightarrow \mathcal{D}(A)$ is the identity functor on $\mathcal{D}(A)$. Therefore, we have $(i_A)^* = I_{\mathcal{D}(A)}$. Thus the cleavage is normalized.

Hence, the functor $P : \mathcal{D} \to \mathcal{A}$ has a normalized split cleavage.

<u>Remark (10)</u> : It can be proved that the functor $f_* : \mathcal{D}(A) \to \mathcal{D}(B)$ is the left adjoint of the functor $f^* : \mathcal{D}(B) \to \mathcal{D}(A)$. Therefore, f_* preserves not only initial object but all colimits.