# CHAPTER - III

## CLEAVAGES AND OPCLEAVAGES IN THE CATEGORY OF R-COMPLEXES

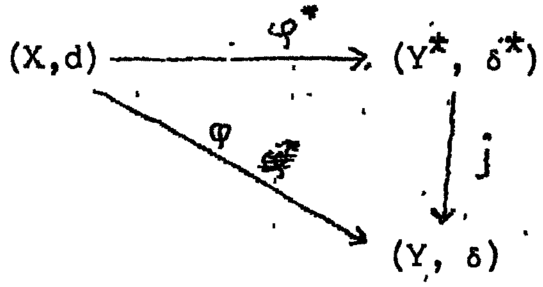
In the previous chapter we have discussed cleavages and opcleavages in the category of R-derivation modules. In this chapter we shall develop analogous concepts for complexes. Since the proofs of the theorems run on same familar lines we have omitted the details.

Definition (3.1) : Let $f : A \longrightarrow B$ be a unitary algebra homomorphism in $\mathcal{A}$. Let $(X,d)$ be an A-complex and $(Y,\delta)$ a B-complex. A graded R-algebra homomorphism $\varphi : X \longrightarrow Y$ is said to be an f - complex homomorphism if and only if $\varphi \mid A = f$ and $\varphi d = \delta \varphi$; and will be denoted as $\varphi : (X,d) \longrightarrow (Y,\delta)$. In this case $(Y,\delta)$ is said to be $\varphi$ - simple if and only if $B \cup \delta(B) \cup \varphi(x)$ generates Y as an R-algebra.

Proposition (3.1) : Let (X, d) be an A-complex. Then for any B-complex $(Y,\delta)$ and an f-complex homomorphism $\varphi : (X,d) \longrightarrow (Y,\delta)$, there exists a $\varphi^*$ - simple B-complex $(Y^*,\delta^*)$ and a B-complex monomorphism

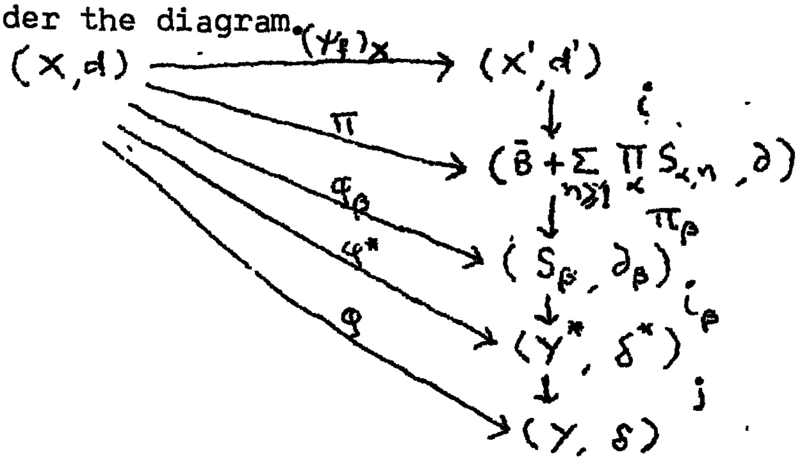
$$j : (Y^*,\delta^*) \longrightarrow (Y,\delta) \text{ such that } j \varphi^* = \varphi.$$

Here $Y^*$ is the R-subalgebra of Y generated by $B \cup \delta(B) \cup \varphi(X)$ and $\delta^* = \delta|Y^*$. $(Y^*,\delta^*)$ is a B-complex. $\varphi^*: (X,d) \longrightarrow (Y^*,\delta^*)$ is defined as $\varphi^* = \varphi$ and $j : Y^* \longrightarrow Y$ is the natural inclusion, and the following diagram commutes.




Proposition (3.2) : For any A-complex (X,d) there exists a B-complex (X',d') and an f-complex homomorphism $(\psi_f)_X$: $(X,d) \longrightarrow (X',d')$ in $\mathcal{C}$ such that for any B-complex $(Y,\delta)$ and any f-complex homomorphism $\varphi : (X,d) \longrightarrow (Y,\delta)$, there exists a unique B-complex homomorphism $\varphi'' : (X',d') \longrightarrow (Y,\delta)$ satisfying $\varphi'' (\psi_f)_X = \varphi$. Moreover (X',d') and $(\psi_f)_X$ are unique in the sense that if there exists another such B-complex $(\bar{X}, \bar{d})$ and an f-complex homomorphism $h_X : (X,d) \longrightarrow (\bar{X},\bar{d})$, then there exists a B-complex isomorphism $I : (X',d') \longrightarrow (\bar{X}, \bar{d})$ satisfying $I\,(\psi_f)_X = h_X$.

Consider the diagram.



Here the B-complex $(Y^*, \delta^*)$, the f-complex homomorphism $\varphi^*$ and the B-complex monomorphism j are as in the previous Proposition (3.1). Let $\{(S_\alpha, \partial_\alpha)\}_{\alpha \in I}$ be a family of $\varphi_\alpha$- simple B-complexes indexed by the set I such that for the $\varphi^*$-simple B-complex $(Y^*, \delta^*)$, there exists a B-complex isomorphism

$i_\beta : (S_\beta, \partial_\beta) \longrightarrow (Y^*, \delta^*)$ for some $\beta \in I$ satisfying $i_\beta \varphi_\beta = \varphi^*$.

Consider the product $(\bar{B} + \sum_{n \geq 1} \pi_\alpha S_{\alpha,n}, \partial)$ of the representative family $\{(S_\alpha, \partial_\alpha)\}_{\alpha \in I}$ of $\varphi_\alpha$ - simple B-complexes and $\pi : (X,d) \longrightarrow (\bar{B} + \sum_{n \geq 1} \pi_\alpha S_{\alpha,n}, \partial)$ the f-complex homomorphism defined as

$$\pi \left(\sum_{n \geq 0} x_n\right) = (\varphi_\alpha(x_0))_\alpha + \sum_{n \geq 1} (\varphi_\alpha(x_n))_\alpha$$

for $\sum_{n \geq 0} x_n$ in X.

$$\pi_\beta : (\bar{B} + \sum_{n \geq 1} \pi_\alpha S_{\alpha,n}, \partial) \longrightarrow (S_\beta, \partial_\beta) \quad \text{the}$$

natural projection is the B-complex homomorphism satisfying $\pi_\beta \pi = \varphi_\beta$. Let X' be the subalgebra of $\bar{B} + \sum_{n \geq 1} \pi_\alpha S_{\alpha,n}$ generated by $\bar{B} \cup \partial(\bar{B}) \cup \pi(X)$. This gives a B-complex $(X',d')$ where $d' = \partial | X'$. Define $(\psi_f)_X : (X,d) \longrightarrow (X',d')$ as $(\psi_f)_X = \pi$. Then $(\psi_f)_X$ is an f-complex homomorphism and

$i : (X',d') \longrightarrow (\overline{B} + \sum_{n\geq 1} \pi_{\alpha} S_{\alpha,n}, \partial)$ denotes the inclusion.

Then obviously $i \, (\psi_f)_X = \pi$.

Let us put $\varphi'' = j \, i_\beta \, \pi_\beta \, i$.
As before, in the above diagram, since all the small triangles commute, the outer-most triangle also commutes and we have
$\varphi'' \; (\psi_f)_X = \varphi$.
$\varphi''$ is unique since $(X', d')$ is $(\psi_f)_X$ - simple. Finally the uniqueness of $(X',d')$ and $(\psi_f)_X$ can be proved on similar lines as in the proof of Proposition (2.2).

Proposition (3.3) : Let $(X,d)$ and $(Y,\delta)$ be A-complexes and $(X',d')$ and $(Y',\delta')$ be the corresponding B-complexes. Then for any A-complex homomorphism $\lambda : (X,d) \longrightarrow (Y,\delta)$ there exists a unique B-complex homomorphism $\lambda' : (X',d') \longrightarrow (Y',\delta')$ such that $\lambda' \; (\psi_f)_X = (\psi_f)_Y \; \lambda$.

Consider the diagram

$$\begin{array}{ccc} (X,d) & \xrightarrow{(\psi_f)_X} & (X',d') \\ \lambda \downarrow & & \downarrow \lambda' \\ (Y,\delta) & \xrightarrow{(\psi_f)_Y} & (Y',\delta') \end{array}$$

The composition $(\psi_f)_Y \, \lambda$ is an f-complex homomorphism. Therefore, Prop (3.2) assures the existence of a unique B-complex homomorphism $\lambda'$ making the above diagram commutative.

Let $f:A \longrightarrow B$ be the R-algebra homomorphism. Define $f_* : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$ as $f_*((X,d)) = (X',d')$ [ as defined in Proposition (3.2)] and $f_*(\lambda) = \lambda'$ [ as defined in proposition (3.3)] for all $(X,d)$ in $\mathcal{C}(A)$ and $\lambda$ in $\mathcal{C}(A)$.

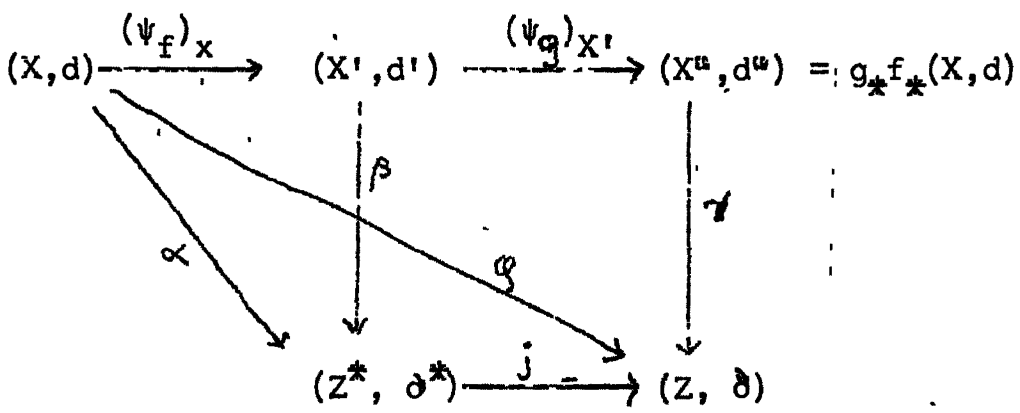
The following Theorem can be proved in the same way in which Theorem (2.1) was proved.

<u>Theorem (3.1)</u> : If $f: A \longrightarrow B$ is an R-algebra homomorphism, then there exists a covariant functor $f_* : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$ defined by $f_*(X,d) = (X',d')$ and $f_*(\lambda) = \lambda'$ for all $(X,d) \in \mathcal{C}(A)$ and $\lambda \in \mathcal{C}(A)$.

<u>Proposition (3.4)</u> : Let A, B, C be commutative unitary R-algebras and $f: A \longrightarrow B$, $g : B \longrightarrow C$ be unitary algebra homomorphisms. Then there exists a natural equivalence

$$c_{fg} : (g\, f)_* \longrightarrow g_*\, f_*.$$

Consider the diagram



Here $(X,d)$ is an A-complex, $(Z,\partial)$ a C-complex and $\varphi : (X,d) \longrightarrow (Z,\partial)$ a g f - complex homomorphism. $(Z^*,\partial^*)$ is a B-complex defined via $g : B \longrightarrow C$ and $j : (Z^*, \partial^*) \longrightarrow (Z, \partial)$ is a g-complex homomorphism and $\alpha : (X,d) \longrightarrow (Z^*, \partial^*)$ is an f-complex homomorphism defined as $\alpha | A = f$ and $\alpha | X_n = \varphi$ for $n \geqslant 1$; satisfying $j \alpha = \varphi$. Since $\alpha$ is an f-complex homomorphism, there exists by Prop(3.2) a unique B-complex homomorphism $\beta : (X',d') \longrightarrow (Z^*, \partial^*)$ such that $\beta (\psi_f)_X = \alpha$. Again $j \beta : (X',d') \longrightarrow (Z,\partial)$ is a g-complex homomomorphism, therefore, there exists by Prop (3.2) a unique C-complex homomorphism $\gamma : (X'',d'') \longrightarrow (Z,\partial)$ such that $\gamma (\psi_g)_{X'} = j \beta$. So we have $\gamma (\psi_g)_{X'} (\psi_f)_X = \varphi$. The uniqueness of $\gamma$ follows from the $(\psi_g)_{X'} (\psi_f)_X$ - simplicity of $(X'',d'') = g_* f_* (X,d)$.

On the other hand, for the g f - complex homomorphism $\varphi : (X,d) \longrightarrow (Z,\partial)$ there exists by Prop (3.2) a unique C-complex homomorphism $\theta : (\overline{X}, \overline{d}) \longrightarrow (Z,\partial)$ such that $\theta \; (\psi_{gf})_X = \varphi$.

$$\begin{array}{ccc} (X,d) & \xrightarrow{(\psi_{gf})_X} & (\overline{X},\overline{d}) = (gf)_* (X,d) \\ & \searrow^{\varphi} & \downarrow \theta \\ & & (Z, \partial) \end{array}$$

Hence by the uniqueness of such C-complexes and C-complex homomorphisms there exists a C-complex isomorphism

$$(c_{fg})_X : (g\ f)_* (X,d) \longrightarrow g_* f_*(X,d)$$

such that $(c_{fg})_X (\psi_{gf})_X = (\psi_g)_{X'} \cdot (\psi_f)_X$.

This shows the existence of a natural equivalence $c_{fg} : (g\ f)_* \longrightarrow g_* f_*$ .

Let $\mathcal{A}$ denote the cateogry of all unitary commutative R-algebras and $\mathcal{C}$ denote the category of all R-complexes. Consider the functor $P : \mathcal{C} \to \mathcal{A}$ defined as

$$P\,(X,d) = X_0 \text{ and } P\,(\varphi) = \varphi_0$$

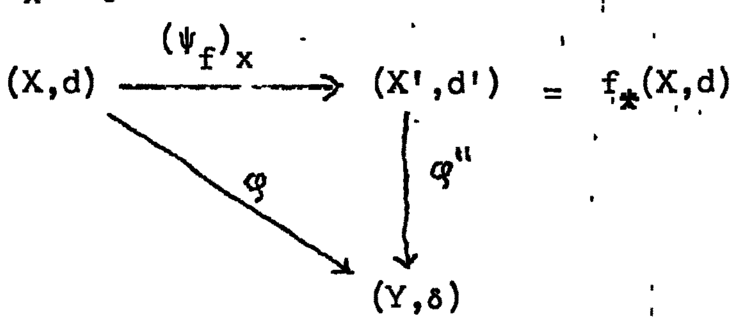
for $(X,d)$ in $\mathcal{C}$ and $\varphi$ in $\mathcal{C}$ .

Then the fibre $P^{-1}(A)$ is the category $\mathcal{C}(A)$ of all A - complexes and A-complex homomorphisms. Let $J_A : \mathcal{C}(A) \to \mathcal{C}$ denote the inclusion functor. Our claim is

<u>Theorem (3.2)</u> : The functor $P : \mathcal{C} \to \mathcal{A}$ admits an opcleavage.

<u>Proof</u> : For each morphism $f:A \to B$ in $\mathcal{A}$ and for any $(X,d)$ in $\mathcal{C}(A)$ there exists a unique $(X',d') = f_*(X,d)$ in $\mathcal{C}(B)$ and an f-complex homomorphism $(\psi_f)_X:(X,d) \longrightarrow f_*(X,d)$ in $\mathcal{C}$ by Prop (3.2). For any morphism $\lambda : (X,d) \longrightarrow (Y,\delta)$ in $\mathcal{C}(A)$, there exists a unique morphism $\lambda' : f_*(X,d) \longrightarrow f_*(Y,\delta)$ in $\mathcal{C}(B)$ by Prop (3.3). Thus each morphism $f:A \longrightarrow B$ in $\mathcal{A}$

gives rise to a functor $f_* : \mathscr{C}(A) \longrightarrow \mathscr{C}(B)$. For each $f : A \longrightarrow B$ in $\mathscr{A}$ there exists a natural transformation $\psi_f : J_A \longrightarrow J_B\, f_*$ satisfying $P\,((\psi_f)_X) = f$ by Prop (3.2). Then for any f-complex homomorphism $\varphi : (X,d) \longrightarrow (Y\ \delta)$ satisfying $P(\varphi) = f$, there exists by Prop (3.2) a unique morphism $\varphi'' : f_*\,(X,d) \longrightarrow (Y,\delta)$ in $\mathscr{C}(B)$ such that $\varphi''\,(\psi_f)_X = \varphi$; i.e. making the following diagram commutative



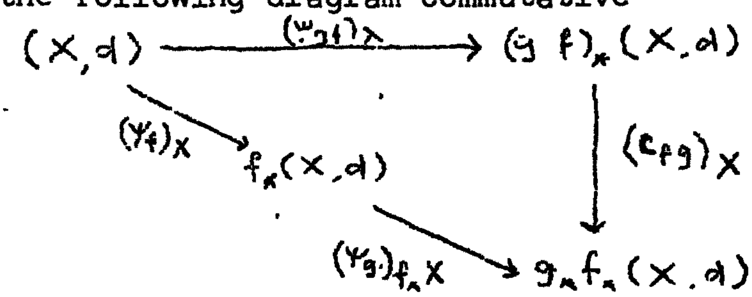

Consider the composition $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathscr{A}$. Then for each $(X,d)$ in $\mathscr{C}(A)$ there exists by Prop (3.4) a uniquely determined morphism

$$(c_{fg})_X : (g\ f)_*\ (X,d) \longrightarrow g_*\ f_*\ (x,d) \quad \text{in } \mathscr{C}(C) \text{ such}$$

that

$$(c_{fg})_X \quad (\psi_{gf})_X = (\psi_g)_{f_* X}\ (\psi_f)_X$$

making the following diagram commutative



It is easily seen that $(c_{fg})_X$ are the components of a natural transformation

$$c_{fg} : (g\ f)_* \longrightarrow g_*\ f_* .$$

Each such $c_{fg}$ is a natural equivalence by Prop (3.4). Thus the functor P: $\mathcal{C} \to \mathcal{A}$ admits an opcleavage $\{f_*, \Psi_f, C_{fg}\}$.

In the following we shall prove that any algebra homomorphism f: A $\longrightarrow$ B in $\mathcal{A}$ gives rise to a covariant functor $f^* : \mathcal{C}(B) \longrightarrow \mathcal{C}(A)$.

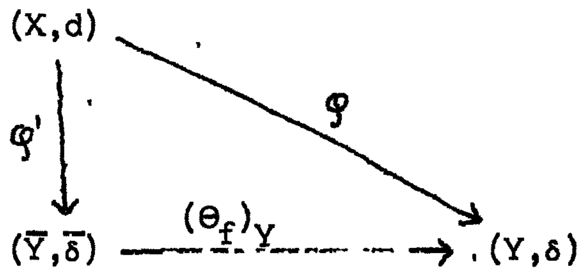
<u>Proposition (3.5)</u> : Let f: A $\longrightarrow$ B be an R-algebra homomorphism. Then for any B-complex $(Y,\delta)$ there exists an A-complex $(\bar{Y},\bar{\delta})$ and the f-complex homomorphism $(\Theta_f)_Y : (\bar{Y}, \bar{\delta}) \longrightarrow (Y,\delta)$.

Here $(Y,\delta)$ is a B-complex. Then $\bar{Y} = A \oplus \sum_{n \geq 1} Y_n$ can be considered as an A-algebra via f: A$\longrightarrow$B. Define $\bar{\delta} : \bar{Y} \to \bar{Y}$ as $\bar{\delta}_o = \delta_o f$ and $\bar{\delta}_n = \delta_n$ for $n \geq 1$.

$\bar{\delta}$ is an R-derivation of degree 1 of $\bar{Y}$ satisfying $\bar{\delta}\,\bar{\delta} = 0$. This gives an A-complex $(\bar{Y}, \bar{\delta})$. Define the mapping $(\Theta_f)_Y : \bar{Y} \to Y$ as $(\Theta_f)_Y | A = f$ and $(\Theta_f)_Y | Y_n$ = identity for $n \geq 1$. Clearly $(\Theta_f)_Y$ is an f-complex homomorphism. Thus with every B-complex $(Y,\delta)$ there exists an A-complex $(\bar{Y},\bar{\delta})$ together with an f-complex homomorphism $(\Theta_f)_Y$.

<u>Proposition (3.6)</u> : Let $(X,d)$ be an A-complex and $(Y,\delta)$ a B-complex. Then for any f-complex homomorphism $\varphi : (X,d) \longrightarrow (Y,\delta)$ there exists a unique A-complex homomorphism $\varphi' : (X,d) \longrightarrow (\bar{Y}, \bar{\delta})$ such that $(\theta_f)_Y \varphi' = \varphi$.

Here define $\varphi' : (X,d) \longrightarrow (\bar{Y},\bar{\delta})$ as $\varphi' \mid A \doteq$ Identity and $\varphi' \mid X_n = \varphi$ for $n \geqslant 1$. Obviously $\varphi'$ is A-complex homomorphism making the following diagram commutative




The uniqueness of $\varphi'$ is obvious.

<u>Proposition (3.7)</u> : Let $(X,d)$ and $(Y,\delta)$ be B-complexes and let $(\bar{X},\bar{d})$ and $(\bar{Y},\bar{\delta})$ be the corresponding A-complexes. Then for any B-complex homomorphism $k:(X,d) \longrightarrow (Y,\delta)$ there exists a unique A-complex homomorphism $\bar{k} : (\bar{X},\bar{d}) \longrightarrow (\bar{Y},\bar{\delta})$ such that $(\theta_f)_Y \bar{k} = k\ (\theta_f)_X$.

The composition $k\ (\theta_f)_X : (\bar{X},\bar{d}) \longrightarrow (Y,\delta)$ is f-complex homomorphism, hence there exists by Prop (3.6) a unique A-complex homomorphism making the following diagram commutative.

$$\begin{array}{ccc} (\overline{X}, \overline{d}) & \xrightarrow{(\theta_f)_X} & (X, d) \\ \bar{k} \downarrow & & \downarrow k \\ (\overline{Y}, \overline{\delta}) & \xrightarrow{(\theta_f)_Y} & (Y, \delta) \end{array}$$

Define a mapping $f^* : \mathcal{C}(B) \longrightarrow \mathcal{C}(A)$ as $f^*((Y,\delta)) = (\overline{Y},\overline{\delta})$ [ as defined in Prop (3.5)] and $f^*(k) = \overline{k}$ [ as defined in Prop (3.7)]. Then the following Theorem can be proved on similar lines as Thm. (2.3).

<u>Theorem (3.3)</u> : If $f: A \longrightarrow B$ is an algebra homomorphism, then there exists a covariant functor $f^* : \mathcal{C}(B) \longrightarrow \mathcal{C}(A)$ defined as $f^*(Y,\delta) = (\overline{Y},\overline{\delta})$ and $f^*(k) = \overline{k}$.

<u>Proposition (3.8)</u> : Let A, B, C be unitary commutative R-algebras and $f:A \longrightarrow B$ and $g : B \longrightarrow C$ be algebra homomorphisms. Then $f^* g^* = (g f)^*$.

For this take a C-complex $(X,d)$. Then $g^*$ associates with $(X,d)$ a B-complex $(\overline{X},\overline{d})$ where $\overline{X} = B \oplus \sum_{n \geq 1} X_n$ and $\overline{d}_o = d_o g$ and $\overline{d}_n = d_n$ for $n \geq 1$.

$f^*$ associates with $(\overline{X},\overline{d})$ an A-complex $(\overline{\overline{X}},\overline{\overline{d}})$ where $\overline{\overline{X}} = A \oplus \sum_{n \geq 1} X_n$ and $\overline{\overline{d}}_o = \overline{d}_o f = d_o g f$ and $\overline{\overline{d}}_n = \overline{d}_n = d_n$ for $n \geq 1$.

Similarly $(gf)^*$ associates with $(X,d)$ an A-complex $(\widetilde{X},\widetilde{d})$ where $\widetilde{X} = A \oplus \sum_{n\geq 1} X_n$ and $\widetilde{d}_o = d_o\, g\, f$ and $\widetilde{d}_n = d_n$ for $n \geq 1$.

$$\begin{array}{ccccc}
f^* g^*(x,d) = (\bar{\bar{X}}, \bar{\bar{d}}) & \xrightarrow{(\theta_f)_{\bar{X}}} & (\bar{X},\bar{d}) = g^*(X,d) & & \\
\Big\downarrow (d_{fg})_X & & & \searrow (\theta_g)_X & \\
(g\, f)^*(X,d) = (\widetilde{X}, \widetilde{d}) & & \xrightarrow{\qquad (\theta_{gf})_X \qquad} & & (X,d)
\end{array}$$

Define the mapping $(d_{fg})_X : (\bar{\bar{X}},\bar{\bar{d}}) \longrightarrow (\widetilde{X},\widetilde{d})$ as $(d_{fg})_X =$

= identity on $\bar{\bar{X}}$. Obviously $(\bar{\bar{X}}, \bar{\bar{d}}) = (\widetilde{X}, \widetilde{d})$ and $(d_{fg})_X$ is

the identity A-complex isomorphism.

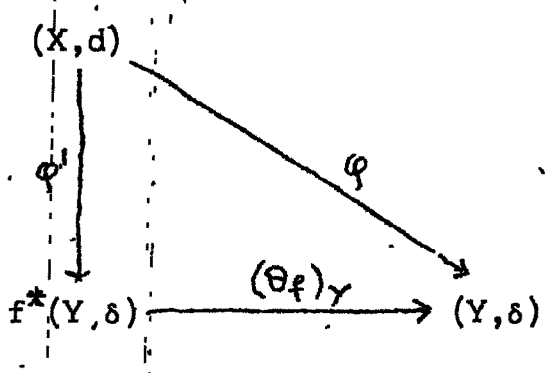
And for any C-complex homomorphism $\varphi : (X,d) \longrightarrow (Y,\delta)$, it is obvious that the following diagram is commutative

$$\begin{array}{ccc}
(\bar{\bar{X}}, \bar{\bar{d}}) & \xrightarrow{(d_{fg})_X} & (\widetilde{X}, \widetilde{d}) \\
f^* g^*(\varphi) = \bar{\bar{\varphi}} \Big\downarrow & & \Big\downarrow \widetilde{\varphi} = (g\, f)^*(\varphi) \\
(\bar{\bar{Y}}, \bar{\bar{\delta}}) & \xrightarrow{(d_{fg})_Y} & (\widetilde{Y}, \widetilde{\delta})
\end{array}$$

Clearly $(d_{fg})_X$ are the components of the identity natural equivalence $d_{fg} : f^* g^* \longrightarrow (g\, f)^*$. Thus $f^* g^* = (g\, f)^*$.

<u>Theorem (3.4)</u> : The functor $P : \mathcal{C} \to \mathcal{A}$ admits a normalized split cleavage $\{f^*, \theta_f, d_{fg}\}$.

<u>Proof</u> : For each $f : A \to B$ in $\mathcal{A}$ and for any $(Y,\delta)$ in $\mathcal{C}(B)$ there exists a unique $(\overline{Y},\overline{\delta})$ in $\mathcal{C}(A)$ and the f-complex homonorphism $(\theta_f)_Y : (\overline{Y}, \overline{\delta}) \to (Y,\delta)$ in $\mathcal{C}$ by Prop. (3.5). For any morphism $k : (X,d) \to (Y,\delta)$ in $\mathcal{C}(B)$ there exists a unique morphism $\overline{k} : (\overline{X},\overline{d}) \to (\overline{Y},\overline{\delta})$ in $\mathcal{C}(A)$ by Prop (3.7). Thus each $f: A \to B$ in $\mathcal{A}$ gives rise to a functor $f^* : \mathcal{C}(B) \to \mathcal{C}(A)$. There exists a natural transformation $\theta_f : J_A \ f^* \to J_B$ satisfying the condition that $P((\theta_f)_Y) = f$ by Prop (3.5). For any f-complex homomorphism $\varphi : (X,d) \to (Y,\delta)$ such that $P(\varphi) = f$, there exists a morphism $\varphi' : (X,d) \to f^*(Y,\delta)$ in $\mathcal{C}(A)$ making the following diagram commutative.

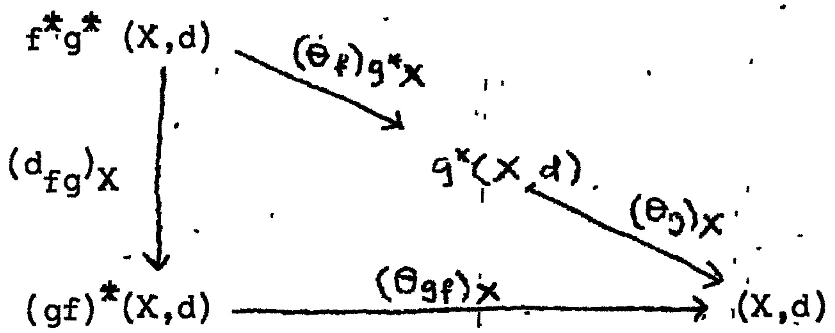



Now consider the composition $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$. Then there exists by Prop (3.8) a uniquely determined morphism.

$$(d_{fg})_X : f^* g^* (X,d) \longrightarrow (g\ f)^*(X,d) \text{ in } \mathcal{C}(A)$$

such that

$(\Theta_{gf})_X (d_{fg})_X = (\Theta_g)_X (\Theta_f)_{g^*X}$. i.e. the following diagram commutes :




$(d_{fg})_X$ are the components of the identiry natural equivalence $d_{fg} : f^* g^* \longrightarrow (g\ f)^*$.

Therefore, the functor P has a split cleavage $\{f^*, \Theta_f, d_{fg}\}$.

Let $I_A : A \longrightarrow A$ be the identity in $\mathcal{A}$, then $(I_A)^* : \mathcal{C}(A) \longrightarrow \mathcal{C}(A)$ is the identity functor on $\mathcal{C}(A)$.

Therefore the cleavage is normalized. Thus the functor $P : \mathcal{C} \rightarrow \mathcal{A}$ has a normalized, split, cleavage.