CHAPTER II

CHAPTER - II

Problems in two dimensional boundary layer theory

1. Introduction to the problem :

E.M.Sparrow [17] studied the boundary layer on a non-isothermal surface with non-uniform free stream velocity. In this paper he obtained an exact solution for thermal boundary layer on a non-isothermal surface subjected to non-uniform free stream velocity is presented in the form of a series and Falkner-Skan type differential equation. By using universal function this equation is solved. H.A.Hassan [6] considered unsteady laminar boundary layers. He shown that solution of Falkner-Skan type equations could be expressed in terms of universal functions. Loitsianskii [8], investigated the universal equations and parametric approximation in the theory of laminar boundary layer. Also he obtained Falkner-Skan type equation.

Freeman and Simpkins [5] generalized the diffusion of species in similar boundary layer with finite recombination rate at the wall and they observed that boundary layer equation reduces to Falkner-Skan type equation and solved by using series solution methods. Mirk [14] made rapid calculations for boundary layer transfer using Wedge solution and asymototic expansion.

Veldman and Vande Vooren [25] obtained a generalized Falkner-Skan equation. They proved that similarity solutions in hydrodynamics could be expressed the differential equation

$$y''' + \mu y y'' + \lambda(1 - y'^2) = 0$$

with the boundary condition

$$y = y' = 0, \text{ at } x = 0$$
$$|y'| \to 1 \text{ as } x \to \infty$$

where λ and μ are real constants with $\mu \neq 0$ and they calculated existence and uniqueness of the above differential equation. Moulden [15] made comments on an exact solution of Falkner-Skan equation when the pressure gradient parameter takes the value $\beta = -1$

Riley [19] studied surface reactions in similar boundary layers and observed that boundary layer equation reduces to Falkner-Skan type equation. William III and Rhyne [26] investigated boundary layer development on a wedge impulsively set into motion. They showed that the solution have been obtained for a number of impulsively started Falkner-Skan type flows ranging from the two-dimensional stagnation point flow to the incipient separation flow. Nanbu [18] established unsteady Falkner-Skan Flow.

Graven and Peletier [4] presented the uniqueness of solution of the Falkner-Skan equation. Craven and Peletier [5] further confirmed solution of Falkner-Skan equation for $\lambda > 1$. Hasting [11] studied Craven and Peletier's [4] problem. Libby and Liu [13] investigated the further solution of Falkner-Skan equation.

Tapas Ranjan Roy [23] studied the boundary layer flow of a power-law of liquid past a wedge in the neighbourhood of the stagnation point. The potential flow velocity in this region is proportional to the arc length raised to a power. The similarity transformation is successfully applied. The asymptotic two point boundary value problem is obtained that is Tapas Ranjan Roy obtained a generalised form of Falkner-Skan equation. This generalised form is then reduced to the initial value problem by applying the Nachtsheim-Swigert iterative scheme and obtained solution. Tapas Ranjan Roy applied the Fourth-order Range-Kutta method to this scheme. The FORTRAN programme for the problem was run over Burroughs 6700 computer for different values of power law index 'n'. The expession for the local non-dimensional skin friction coefficient, displacement thickness, momentum thickness and kinetic energy thickness were found.

M.A.Abdel-Gaid, M.A.Khaider and M. Elangery [1] obtained the solution of boundary layer equation for a Non-Newtonian electrically conducting power law fluid past semi-infinite porous plate. They calculated the thickness of boundary layer. They applied the successive approximation (Pai 1956, Nasiki, 1965) method to the differential equation and they found that the successive approximation method gave good results with zero approximation.

In this Chapter in Section (3) we studied the boundary layer flow of a power-law of liquid past a flat plate and obtained a generalized Blasius differential equation, by similar argument made by Tapas Ranjan Roy (23) in his paper and we discussed the method of solving this differential equation by using Nacutsheim-Swigert (1965) iterative scheme and forth-order Range-Kutta method. In Section (4) of this chapter we studied the boundary layer flow of a power-law of convergent channel, by applying the same above technique we can solve differential equation. Before our investigation we require some boundary layer equations with their solutions and some basic concepts as pre-requisite which have been given below in Section (2a) and Section (2b) respectively.

(2a) Basic boundary layer equations with their solutions:

(I) Similar solutions of the boundary layer equations :

Blasisu [3] and Topfer [24] obtained similar solutions of the boundary layer equations. They considered u/U is a function of η, i.e. $u/U = f(\eta)$ where $\eta \sim y/\delta$. η is called as similarity variable and $u/U(x)$ is the function of one variable

$$\therefore \quad \frac{u}{U(x)} = f\left(\frac{y}{g(x)}\right), \text{ where } g(x) \sim \delta \text{ where } \delta \text{ is}$$

the boundary layer thickness. Hence for any two streamwise co-ordinates x_1 and x_2

$$\frac{u(x_1,\ y/g(x_1))}{U(x_1)} = \frac{u(x_2,\ y/g(x_2))}{U(x_2)} \quad \ldots (1)$$

Similar solutions will be possible when partial differential equations transformed into ordinary differential equations. For limited choice of $U(x)$ and the corresponding form of $g(x)$, the similar solutions exists. This problem was first studied by Goldstein [8] and later improved by Mangter [16]. The boundary layer equations for a incompressible flow are

$$u \frac{\partial u}{\partial x} + \frac{v \partial u}{\partial y} = \frac{du}{dx} + \gamma \frac{\partial^2 u}{\partial y^2} \quad \ldots (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad \ldots (3)$$

with the boundary conditions.

$$y = 0; \; u = v = 0$$

$$y \longrightarrow \infty \; ; \; u = U(x) \quad \ldots (4)$$

We introduce stream function $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y}, \; v = -\frac{\partial \psi}{\partial x} \quad \ldots (5)$$

Then the equation of continuity is satisfied and equation of motion becomes

$$\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial y^2} = U \frac{dU}{dx} + \gamma \frac{\partial^3 \psi}{\partial y^3} \quad \ldots (6)$$

The corresponding boundary conditions are

$$y = 0; \; \psi = \frac{\partial \psi}{\partial y} = 0$$

$$y \longrightarrow \infty ; \; \frac{\partial \psi}{\partial y} = U(x) \quad \ldots (7)$$

Now we shall introduce a non-dimensional quantities

$$\xi = x/L, \quad \eta = y \frac{\sqrt{Rel}}{L.g(x)}$$

$$f(\xi, \eta) = \frac{\psi(x,y).\sqrt{Rel}}{L.U(x)\,g(x)} \qquad \ldots (8)$$

$Rel = \dfrac{U_\infty L}{\nu}$ = Reynold number where U_∞ is the characteristic velocity of flow

L is the characteristic length of flow.

g(x) is the non-dimensional function of x.

From (8) we calculate ,

$$u = \frac{\partial \psi}{\partial y} = U f' \qquad \ldots (9)$$

$$-V = \frac{\partial \psi}{\partial x} = \frac{L}{\sqrt{Rel}} \left[f \frac{d}{dx} (Ug) + Ug \left(\frac{1}{L} \frac{\partial f}{\partial \xi} + \frac{f' \partial \eta}{\partial x} \right). \right] \qquad \ldots (10)$$

$$\frac{\partial u}{\partial x} = f' \frac{\partial u}{\partial x} + U \left(\frac{1}{L} \frac{\partial f'}{\partial \xi} + f'' \frac{\partial \eta}{\partial x} \right) \qquad \ldots (11)$$

$$\frac{\partial u}{\partial y} = \frac{U f'' \sqrt{R_e l}}{L.g(x)} \qquad \ldots (12)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U R_e l}{L^2 g^2} f''' \qquad \ldots (13)$$

Substituting equations (9) to (13) in equation (2), we get

$$f''' + \alpha ff'' + \beta (1-f'^2) = \frac{U g^2}{U_\infty} \left(f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi} \right) \quad \ldots (14)$$

$$\left.\begin{aligned} \text{where } \alpha &= \frac{Lg}{U} \frac{d}{dx} (Ug) \\ \beta &= \frac{Lg^2}{U} \frac{dU}{dx} \end{aligned}\right) \quad \ldots (15)$$

The corresponding boundary conditions are

$$\left.\begin{aligned} \eta = 0;\ f = 0;\ f' = 0 \\ \eta \rightarrow \infty;\ f' = 1 \end{aligned}\right) \quad \ldots (16)$$

Since similar solutions exist if f and f' are independent of ξ i.e. they are functions of η only. Therefore α, β are independent of x, i.e. α, β are constants. With this arguments (14) and (16) are reduced to

$$f''' + \alpha ff'' + \beta (1 - f'^2) = 0 \quad \ldots (17)$$

with the boundary conditions.

$$\left.\begin{aligned} \eta = 0;\ f = 0,\ f' = 0 \\ \eta \rightarrow \infty;\ f' = 1 \end{aligned}\right) \quad \ldots (18)$$

Equation (17) with (18) is known as Falkner and Skan (7) equation and solutions have been studied in detail.

Now we shall find out the forms of U(x) and g(x) for which similar solution exists. From (15) we find that

$$2\alpha - \beta = \frac{L}{U_\infty} \frac{d}{d} (g^2 U) \quad \ldots (19)$$

Case I : $2\alpha = \beta \neq 0$

Then from equoation (19) we get

$$g^2 \frac{U}{U_\infty} = (2\alpha - \beta) \ x/L \quad \ldots (20)$$

when $x = 0$, $g^2 \frac{U}{U_\infty} = 0$.

gives Blasisus and Hiemenz solutions.

Due to equation (15), equation (20) becomes

$$\alpha - \beta = \frac{U_g L}{U_\infty} \frac{dg}{dx} \quad \ldots (21)$$

Simplifying equation (20) we get

$$\left[\frac{U(x)}{U}\right]^{\alpha-\beta} = C\, g^{\beta} \quad \ldots (22)$$

where C is a constant of Integration. Eliminating g between equation (20) and (22) we get

$$\frac{U(x)}{U_\infty} = C^{2/(2\alpha-\beta)} \left[(2\alpha - \beta)\frac{x}{L}\right]^{\beta/(2\alpha-\beta)} \qquad \ldots (23)$$

Due to equation (20), equation (23) becomes

$$g(x) = \left[(2\alpha - \beta)\frac{U_\infty}{U}\frac{x}{L}\right]^{\frac{1}{2}} \qquad \ldots (24)$$

Hence from equations (8) and (24) the similarity variable will be given by

$$\eta = y\left[\frac{1}{(2\alpha - \beta)}\frac{U}{\nu x}\right]^{\frac{1}{2}} \qquad \ldots (25)$$

Hence equation (23) gives the potential flow velocity distribution for which the similar solution exists and the function g(x), which is proportional to the boundary layer thickness by equations (24) and (25) respectively. The equation (23) can be normalized by taking the following considerations :

(1) If $\alpha \neq 0$

Let $\alpha = 1$ and $m = \dfrac{\beta}{2-\beta}$

Then the equations (23) and (25) in the normalized form may be written as

$$\frac{U(x)}{U_\infty} = C^{1+m} \left(\frac{2}{1+m} \cdot \frac{x}{L} \right)^m \qquad \ldots (26)$$

$$g(x) = \left(\frac{2}{1+m} \frac{U_\infty}{U} \cdot \frac{x}{L} \right)^{\frac{1}{2}} \qquad \ldots (27)$$

$$\eta = y \left(\frac{1+m}{2} \cdot \frac{U}{\nu x} \right)^{\frac{1}{2}} \qquad \ldots (28)$$

Thus from the above equation we can write the potential flow velocity varies as some power of x i.e.

$$U(x) \sim x^m$$

and similar solution exists.

Examples of potential flow is given by equation (26) include the following :

$\alpha = 1, \ \beta = 0; \quad U(x) = \text{constant}$

flow past a flat plate

$\alpha = 1, \ \beta = 1, \ U(x) \sim x$

flow near the forward stagnation point.

$\alpha = 1, \ 0 < \beta < 2 ; \quad U(x) \sim x^{\beta/(2-\beta)}$

flow past a wedge with wedge angle $\pi \beta$.

(2) If $\alpha = 0$

Then from equation (23) we see that $U(x)$ is proportional to $1/x$ for all values of β; the equation (23) may be normalised by taking $\beta = \pm 1$. This is a case of a two-dimensional source or sink as $U(x)$ is positive or negative. It may be interpreted as flow in divergent or convergent channel with flat walls

$$\alpha = 0,\ \beta = 1;\ U(x) = -\frac{u_1}{x} \quad (u_1 > 0)$$

flow in a convergent channel.

$$\alpha = 0,\ \beta = -1;\ U(x) = \frac{u_1}{x} \quad (u_1 > 0)$$

flow in a divergent channel.

<u>Case II</u> : If $2\alpha - \beta = 0$.

From equation (19) we find that

$$g^2\ U(x) = \text{Constant} \qquad \ldots (29)$$

from second equation in (15) and form from equation (29) we get

$$\frac{1}{U}\ \frac{dU}{dx} = \text{Constant}$$

$$\text{i.e.}\quad U(x) = e^{px}$$

where p is positive constant when $\beta \rightarrow 2$, then $m \rightarrow \infty$. Then the solution is taken as the limiting form the Case (I).

(II) Boundary layer flow past a wedge :

A boundary layer flow of incompressible fluid past a wedge, with the wedge angle $\pi\beta$ where $\beta = 2m/(m+1)$ as shown in Fig.1.

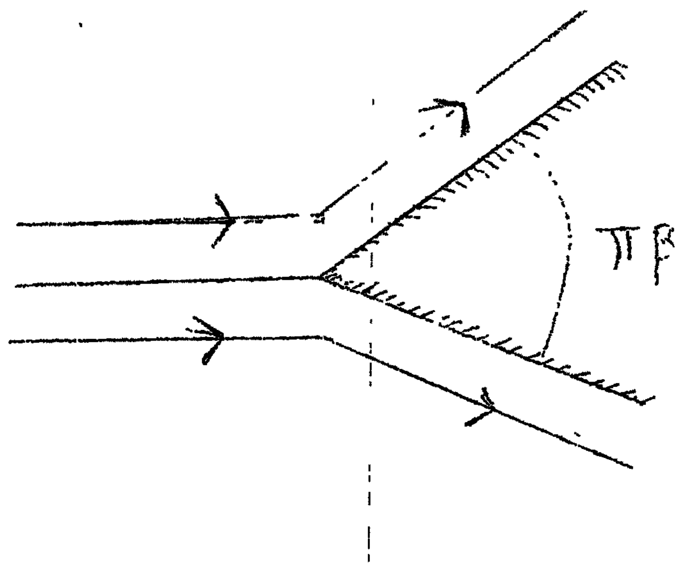

Fig.1. Boundary layer fluid past a wedge for incompressible flow.

Let origin be taken at the stagnation point. The x-axis along the wall and the y-axis perpendicular to the wall. Thus we have the boundary layer equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{du}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \qquad \ldots(1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad \ldots(2)$$

with the boundary conditions

$$y = 0; \quad u = 0 = v$$

$$y \to \infty; \quad u = U(x) \qquad \ldots (3)$$

where ν is the kinematic viscosity and $U(x)$ is the potential flow velocity. Now we introduce the stream function $\psi(x,y)$, which satisfies the continuity equations (2)

Thus we take

$$u = \frac{\partial \psi}{\partial x} \qquad v = -\frac{\partial \psi}{\partial y} \qquad \ldots (4)$$

by using equation (4), equation (1) takes the form as

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = U\frac{dU}{dx} + \nu\frac{\partial^3 \psi}{\partial y^3} \quad ---(5)$$

The corresponding boundary conditions are

$$y = 0; \quad \psi = \frac{\partial \psi}{\partial y} = 0$$

$$y \to \infty, \quad \frac{\partial \psi}{\partial y} = U(x) \qquad \ldots (6)$$

In the neighbourhood of the stagnation point the potential flow velocity U may be written as

$$U(x) = u_1 x^m, \; m \geq 0, \quad m = \frac{\beta}{2-\beta}$$

Hence by the substituting this values of $U(x)$ in (8) of the previous articles, η takes the form -

$$\eta = y \left[\frac{1 + m}{2} \; \frac{U(x)}{x} \right]^{\frac{1}{2}}$$

$$\eta = y \left[\frac{1 + m}{2} \; \frac{u_1}{\gamma} \right]^{\frac{1}{2}} \; x^{\frac{(m-1)}{2}} \qquad \ldots (7)$$

and $$\psi(x,y) = \left(\frac{2 \gamma u_1}{m + 1} \right)^{\frac{1}{2}} x^{\frac{(m+1)}{2}} \; f(n). \qquad \ldots (8)$$

Using equations (7) and (8) we get the following

$$u = \frac{\partial \psi}{\partial y} = U(x)\, f'(\eta) = u_1 \; x^m f' \; (\eta) \qquad \ldots (9)$$

$$v = - \frac{\partial \psi}{\partial x} = - \left(\frac{m + 1}{2} \cdot \gamma u_1 \right)^{\frac{1}{2}} x^{\frac{(M-1)}{2}} \left[f(\eta) + \right.$$

$$\left. + \frac{m-1}{m+1} \; y\, f'(\eta) \right] \qquad \ldots (10)$$

$$u \frac{\partial u}{\partial x} = m u_1^2 \; f'^2 \; x^{(2m-1)} \; +$$

$$+ u_1^2 \frac{(m-1)}{2} \; x^{\frac{(5m-3)}{2}} \; f'.f''.y \left[\frac{(1+m)}{2} \; u_1 \right]^{\frac{1}{2}} \ldots (11)$$

$$v \frac{\partial v}{\partial y} = - \left(\frac{m+1}{2} \right) u_1^2 \; x^{(2m-1)} \; ff'' \; -$$

$$- \frac{(m-1)}{2} \, u_1^2 \left[u_1 \frac{(1+m)}{2} \right]^{\frac{1}{2}} y \; x^{\frac{(5m - 3)}{2}} \; ff'' \qquad \ldots (12)$$

$$U \frac{du}{dx} = m\, u_1^2 \; x^{(2m-1)} \qquad \ldots (13)$$

$$\nu \frac{\partial^2 u}{\partial y^2} = \left(\frac{1+m}{2} \right) u_1^2 \; x^{(2m-1)} \; f''' \qquad \ldots (14)$$

Due to equations (9) and (14) the equation (1) takes the form

$$f''' + ff'' + \frac{2m}{1+m} (1 - f'^2) = 0$$

$$\text{i.e. } f''' + ff'' + \beta (1 - f'^2) = 0 \qquad \ldots (15)$$

$$\text{where } \beta = \frac{2m}{1+m}$$

subjects to the boundary conditions

$$\eta = 0; \; f = 0 = f'$$

$$\eta \to \infty ; \quad f' = 1 \qquad \ldots (16)$$

Equation (15) with equation (16) is known as the Hartree's [9] equation. It is a particular case of Falkner and Skan equation. The numerical solution of equation (15) was first investigated by Hartree [9] and later by other workers.

Hartree's Solution :

$$\text{Let } f(\eta) = \sum_{n=2}^{\infty} \frac{a_n \eta^n}{n!} \qquad \ldots (17)$$

which satisfies the first two boundary conditions of (16) substitution (17) in equation (15) we find the coefficient expressed in terms of $a_2 = \alpha$.

$$a_2 = \alpha$$
$$a_3 = -\beta$$
$$a_4 = -(1 - 2\beta)\,\alpha^2$$
$$a_6 = 2\,(1 - 3\beta)\,\alpha\beta$$
$$a_7 = -2\,(2 - 3\beta)\,\beta^2$$
$$a_8 = -(1 - 2\beta)\,(11 - 10\beta)\alpha^3$$
$$a_9 = -2\,(45 - 11\beta + 66\,\beta^2)\,\beta\alpha^2$$
$$a_{10} = 16\,(2-3\beta)(8 - 7\beta)\,\alpha\beta^2$$

and so on.

It may be noted that $\alpha = f''(0)$ is still unknown.

For finding value of α, Meksyn (17) adopted the following procedure. By substituting the values of f and f' from equation (17) in (15) we get a linear differential equation in $f''(\eta)$ $\therefore f''(\eta) = e^{-F(\eta)}\phi(\eta)$ ... (18)

where $F(\eta) = \int_0^\eta f(\eta)\, dy$ ... (19)

and $\phi(\eta) = \alpha - \beta \int_0^\eta (1 - f'^2)\, e^{F(\eta)}\, d\eta$ ... (20)

equation (18) satisfies the boundary condition at $\eta = 0$. Then unknown parameter α is found from the equation

$$\bar{e}^{F(n)} \phi(n)\, d\eta = 1 \qquad \ldots (21)$$

Hartree (9) solved the equation (15) numerically on a differential analyser for different values of β: From the solution of (15) we obtained boundary layer parameters.

(a) Displacement thickness :

$$\delta_1 = \int_0^\infty (1 - \frac{u}{v})\, dy$$

$$\delta_1 = \left(\frac{2\gamma}{1+m} \frac{x^{1-m}}{u_1} \right)^{\frac{1}{2}} . A(B)$$

$$\text{where } A(B) = \int_0^\infty (1 - f')\, d\eta$$

$$\text{and } A(B) = \lim_{n \to \infty} (\eta - f(\eta))$$

(b) Momentum thickness :

$$\delta_2 = \int_0^\infty \frac{u}{U} (1 - \frac{u}{U})dy$$

$$= \left(\frac{2\gamma}{1+m} \frac{x^{1-m}}{u_1} \right)^{\frac{1}{2}} B(\beta)$$

$$\text{where } B(\beta) = \int_0^\infty f' (1 - f')\, d\eta$$

$$= f'' (0) - \beta A(B) - \beta B(\beta).$$

(c) Shearing Strees :

$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right) y = 0$$

$$= \mu u_1 \left(\frac{m + 1}{2 \gamma} \cdot u_1 \; x^{(3m-1)} \right)^{\frac{1}{2}} f''(0)$$

(III) Boundary layer flow along the wall of a convergent channel of incompressible flow

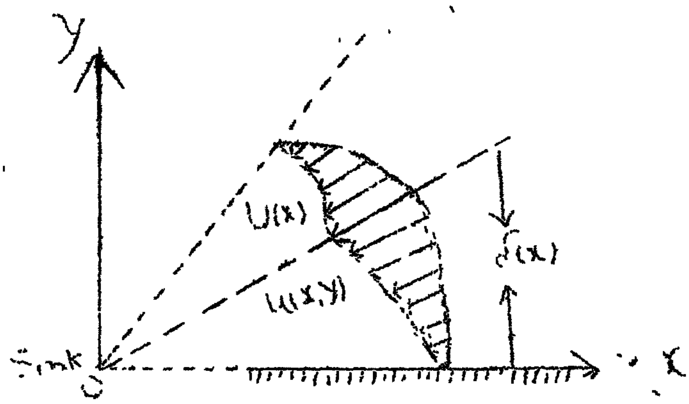

Fig. 2 : Boundary layer flow along the wall of a convergent channel of incompressible flow.

The potential flow velocity is given by

$$U(x) = \frac{-u_1}{x} \qquad (u_1 > 0)$$

where x is measured along the wall of the channel, and u_1 is the strength of the sink placed at O. Y - axis is perpendicular to the wall. This is an another example in which a similar solution of the boundary layer equations is possible. Pholhausen, K. (1921) was the first who obtained the solution of this problem. For this he introduced the similarity variable.

$$\eta = \frac{y}{x} \left(\frac{u_1}{\nu} \right)^{\frac{1}{2}} \qquad \ldots (1)$$

$$= - (\nu \, u_1)^{\frac{1}{2}} f(n) \qquad \ldots (2)$$

from equation (1) and (2) the equation of the boundary layer (1) in artical (2) takes the forms as

$$f''' + 1 - f'^2 = 0 \qquad \ldots (3)$$

with the boundary conditions

$$\left.\begin{aligned} &\eta = 0; \quad f = 0, \quad f' = 0 \\ &\eta \to \infty \; ; \quad f' = 1 \end{aligned}\right\} \qquad \ldots (4)$$

For the solution of equation (3) we multiplying it by 2f" and integrating we get

$$f''^2 - \frac{2}{3} (f' - 1)^2 (f' + 2) = C \qquad \ldots (5)$$

where C is constant of integration.

When $\eta = 0$; $f' = 0$, $f = 0$

$n = \infty$; $f' = 1$, and $f'' = 0$.

Hence $C = 0$

equation (5) becomes

$$f''^2 - 2/3\,(f' - 1)^2\,(f' + 2) = 0 \qquad \ldots (6)$$

$$f'' = \left[2/3\,(f' + 2)\right]^{\frac{1}{2}}\,(1 - f')$$

Let $f' + 2 = 3 \tan h^2 \zeta$ and integrate it

we find that

$$\zeta = \frac{\eta}{(2)^{\frac{1}{2}}} + K$$

where K is constant of integration, determined from the boundary conditions.

$$\eta = 0; \quad f' = 0$$

$$K = \tan h^{-1}\,(2/3)^{\frac{1}{2}} \quad = 1.146$$

Hence $u/U = f'(\eta) = 3 \tan h^2 \zeta - 2$

$$f' = 3 \tan h^2 \left(\frac{\eta}{(2)^{\frac{1}{2}}} + 1.146 \right) - 2$$

<u>Boundary layer parameters</u>

(1) <u>Displacement thickness</u> :

$$\delta_1 = \int_0^\infty (1 - u/U)\,dy$$

$$= 0.778x\,\left(\nu/u_1 \right)^{\frac{1}{2}}$$

(2) Momentum thickness :

$$\delta_2 = \int_0^\infty u/U \,(1 - u/U)\, dy$$

$$= 0.376\, x\, (\nu/u_1)^{\frac{1}{2}}$$

(3) Shearing Stress :

$$T_w = -\mu \left(\frac{\partial u}{\partial y}\right) y = 0$$

$$T_w = -1.54\, \mu\, \frac{u_1}{x^2}\, (u_1/\nu)^{\frac{1}{2}}.$$

3 ...) Basic Concepts :

(I) Newtonian Fluid :

Newton observed that in a simple rectilinear motion of fluid two neighbouring fluid layers, one is moving over the other with the same relative velocity, will experience a tangential force proportional to the relative velocity between the two layers and inversely proportional to the distance between the layers. If the two neighbouring fluid layers are moving with velocities u and u + δu and at a distance δy the shearing stress is given by

$$\tau \propto \frac{\delta u}{\delta y} \quad \text{or} \quad \tau = \mu \frac{du}{dy}$$

This is called "the Newtonian hypothesis" and a fluid satisfying this hypothesis is called Newtonian Fluid.

The constant of proportionality μ is called as coefficient of viscosity and du/dy is the strain-rate of the fluid.

Newtonian fluids are also called as viscous fluids.

(II) Non-Newtonian Fluids :

The fluids which do not satisfy the Newtonian hypothesis that fluids are called as non-Newtonian fluids.

The non-Newtonian fluids are classified into following three ways :

(i) Purely viscous fluids

(ii) Visco-plastic fluids and perfectly plastic materials.

(iii) Visco-elastic fluids.

(i) Purely Viscous Fluids :

A fluid in which the stress tensor p_{ij} is given as the function of the strain rate is called purely viscous fluid; provided in the absence of strain-rate the stress tensor is zero. Mathematically, this can be given as

$$P_{ij} = f(e_{ij}) \text{ and } f(0) = 0$$

where the function of cannot be completely arbitrary. The different forms of the function f gives different types of purely viscous fluids.

Types of purely viscous fluids :

(a) Reiner-Rivlin Fluids :

A fluids satisfying the following constitutive equation is called Reiner-Rivlin Fluids :

$$T_{ij} = - p\delta_{ij} + 2\mu e_{ij} + \mu_1 e_{ik} e_{kj} + \mu_2\delta_{ij}$$

where $T_{ij} = - p\delta_{ij} + p_{ij}$

μ, μ_1, μ_2 are constants.

δ_{ij} - Kroneker delta tensors.

(b) Power law of fluids :

If the function f in the equation

$p_{ij} = f(e_{ij})$ and $f(0) = 0$

is taken to be the analytic function of the arguments and we can expand it as power series in strain-rate tensor i.e.

$$p_{ij} = 2\mu e_{ij} + \mu_1 e_{ik} e_{kj} + \mu_2 e_{il} e_{lk} e_{kj} + \dots$$

The relation between the stress tensor p_{ij} and strain-rate tensor is in which stress tensor is proportional to some power (may be fractional) of the strain-rate tensor i.e.

$$p_{ij} \propto (e_{ij})^n \text{ or } p_{ij} = \quad (e_{ij})^n$$

where $\mathcal{C}$ is the constant; since p_{ij} is second order tensor, hence right hand side should also be a tensor of second order.

We can write the above equation as

$$p_{ij} = 2\mu \, (2\, e_{lm}\, e_{lm})^{\frac{n-1}{2}}$$

where μ is called the coefficient of viscosity and n is called the index of the power law of fluids.

If $n = 1$ then Newtonian fluids

If $n > 1$ then pseudo-plasic fluids

If $n < 1$ dilatant fluids.

(ii) Visco plastic and perfectly plastic materials :

If we apply a certain shearing stress on viscous fluids, It may cause a continuous deformation in the fluids. But in the case of materials like paints, pastes etc. we find that if we apply a shearing stress less than a certain quantity the material does not move at all. But when this shearing stress exceeds the material starts moving and strain-rate of material depends upon the applied stress. Such a materials are called plastics.

The constitutive equation for viscoplastic is given by

$$p_{ij} = e_{ij}\ (2\mu + T_0/\tilde{V}_2^{\frac{1}{2}})$$

where $T_2^{\frac{1}{2}} > T_0$ where $T_2 = T_0 + 2\mu\tilde{V}_2^{\frac{1}{2}}$ and $e_{ij} = 0$

when $T_2^{V2} \leq T_0$ where μ is the coefficient of viscosity, T_0 is the yield stress gives the solid character.

If $\mu \longrightarrow 0$ then the material is called as perfectly plastic material.

(iii) Visco-elastic fluids :

The fluids in which the stress depends upon the rate of deformation and when stress is removed the strain-rate becomes zero. But the deformation has accumulated persists, that is it forgets its original position. But there are some fluids like soap solution, polymers have some elastic properties. This fluids is called as visco-elastic fluids.

Now with this backward we study the boundary Layer of a power law liquid past a flat plate and along the wall of a convergent channel.

4 3. Boundary layer flow of a power law liquid past a flat plate :

(a) Introduction :

The boundary layer flow of power-law liquid past a flat plate is studied in the neighbourhood of the stagnation point. The potential flow velocity in this region is constant every where. The similarity transformation is successfully applied and we get the generalized form of the Blasius equation for a boundary layer flow of a power-law liquid past a flat plate.

...

(b) Basic equations :

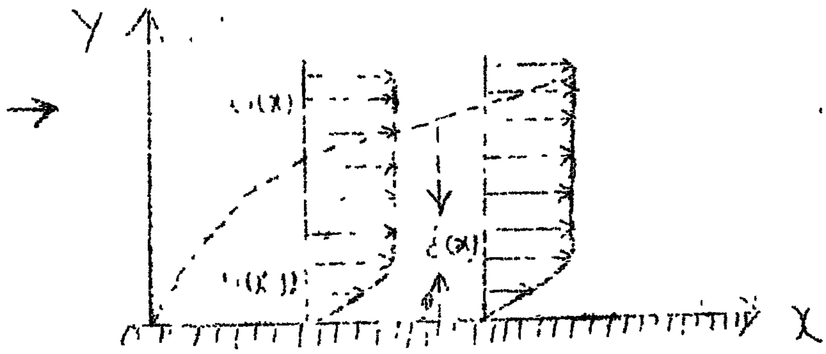

Fig.3 : Boundary layer on a flat plate

A boundary layer flow of a power law liquid past a flat plate is shown as in the Fig.1. is the generalization of the problem for ordinary Viscous liquid. Let origin be taken at the stagnation point. Let x - axis be along the flat plate and y - axis perpendicular to the flat plate and the potential flow velocity U be constant every where in the direction of the x-axis.

Then we have the boundary layer equation in the form as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = K \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right)^n - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad \ldots (1)$$

$$\frac{\partial p}{\partial y} = 0 \quad \ldots (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \ldots (3)$$

with the boundary conditions :

$$\left.\begin{array}{ll} u = v = 0; & y = 0 \\ u = U(x); & y = \infty \end{array}\right) \qquad \ldots (4)$$

where, K is kinematic viscosity for the power-law liquid, n is the power law index and U(x) is the local potential flow velocity.

We can assume that

$$U\frac{dU}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

with this equation (1) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + K\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n \qquad \ldots (5)$$

But for a flat plate we have

$$\frac{dU}{dx} = 0$$

Hence equation (5) becomes

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = K\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n \qquad \ldots (6)$$

with the boundary conditions,

$u \;\partial$

$$\left.\begin{array}{ll} u = v = 0; & y = 0 \\ u = U(x); & y = \infty \end{array}\right) \qquad \ldots (7)$$

for the solution of equation (6) with the boundary condition (7) we take the stream function ψ which satisfies the continuity equation, that is we take

$$u = \frac{\partial \psi}{\partial y}\ , \quad v = -\frac{\partial \psi}{\partial x} \qquad \ldots (8)$$

Due to equation (8), equation (6) becomes

$$\psi_y \, \psi_{yx} - \psi_x \, \psi_{yy} = K \frac{\partial}{\partial y} (\psi_{yy})^n \qquad \ldots (9)$$

with the boundary conditions

$$\left.\begin{array}{ll} \psi_y = \psi_x = 0; & y = 0 \\ \psi_y = U(x); & y = \infty \end{array}\right) \qquad \ldots (10)$$

In the neighbourhood of the stagnation point the potential flow velocity U may be written as

$$U = C$$

For the similar solution, we use the following transformation with the non-dimensional variables

...

$$f(\eta) = \frac{\psi(x,y)}{(n+1)^{\frac{1}{(n+1)}} \; C^{\frac{(2n-1)}{(n+1)}} \; K^{\frac{1}{(n+1)}} \; x^{\frac{1}{(n+1)}}} \quad \ldots (11)$$

$$\text{and} \quad \eta = \frac{y.\left(\frac{1}{(n+1)}\right)^{\frac{1}{(n+1)}} \; C^{\left(\frac{2-n}{n+1}\right)} \; x^{-\frac{1}{(n+1)}}}{K^{\frac{1}{(n+1)}}} \quad \ldots (12)$$

From equations (11) and (12) we obtained

$$u = \psi_y = Cf' \quad \ldots (13)$$

$$u \frac{\partial u}{\partial x} = \psi_y \cdot \psi_{yx}$$

$$= \frac{-1 \; C^{\left(\frac{n+4}{n+1}\right)} \; y.\, f'f'' \; x^{-\frac{(n-2)}{(n+1)}}}{(n+1)^{\frac{1}{(n+1)}} \; (n+1).\, K^{\frac{1}{(n+1)}}} \quad \ldots (14)$$

$$V = -\psi_x$$

$$= \frac{-(n+1)^{\frac{1}{(n+1)}} \; C^{\left(\frac{2n-1}{n+1}\right)} \; K^{\frac{1}{(n+1)}} \; f \; x^{\frac{-n}{(n+1)}}}{(n+1)} +$$

$$+ \frac{1}{n+1} \,.\, C\, y\, f'\, x^{-1}$$

$$V \frac{\partial u}{\partial x} = -\psi_x \psi_{yy}$$

$$= \frac{-C^2\, ff''\, x^{-1}}{(n+1)} + \frac{C^{\frac{(n+4)}{(n+1)}}\; yf'f''\, x^{\frac{-(n-2)}{(n+1)}}}{(n+1)^{\frac{1}{(n+1)}}\,(n+1).\; K^{\frac{1}{(n+1)}}} \quad \ldots(15)$$

$$K \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n = K \frac{\partial}{y}\left(\psi_{yy}\right)^n$$

$$= \frac{n\, C^2\, x^{-1}\, f'''\, (f'')^{n-1}}{(n+1)} \quad \ldots (16)$$

By using equations (14), (15) and (16) equations (9) and (10) take the form

$$nf'''\,(f'')^{n-1} + ff'' = 0 \quad \ldots (17)$$

with the boundary conditions.

$$\left.\begin{array}{ll} f' = f = 0; & \eta = 0 \\ f' \longrightarrow 1; & n \longrightarrow \infty \end{array}\right\} \quad \ldots (18)$$

For n=1 equation (17) becomes the Blasius equation for a flat plate.

For the solution of equation (17) we can use the Nachtsheim-Swigert iterative scheme (1965) and fourth-order Range-Kutta method. Also we can use the FORTRAN programme for this problem which can be run over Burroughs 6700 computer for different values of the power law index n. Thus we can determine the local non-dimensional skin friction coefficient, displacement thickness, momentum thickness and kinetic energy thickness. This is under our investigation.

4. The boundary layer flow of a power law liquid along the wall of a convergent channel.

(a) Introduction : A boundary layer flow of oower law liqud liquid along the wall of a convergent channel is studied in the neighbourhood of a sink. The potential flow velocity in this region is inversely proportional to the arc length. By using the similarity transformation we can find the generalized equation for the convergent channel.

(b) Basic equations :

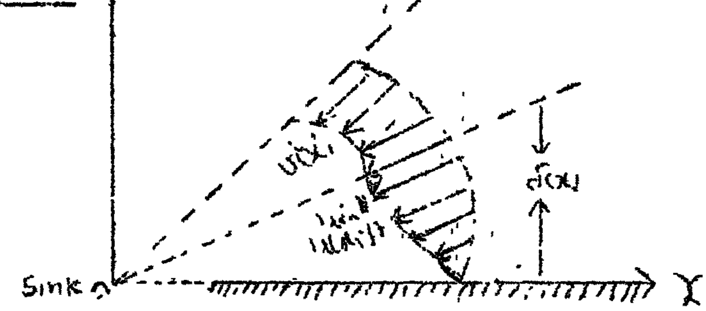

Fig.4 : A boundary layer flow of a power law liquid past along the wall of the convergent channel.

A boundary layer flow of a power law liquid past along the wall of the convergent channel is a generalization of similar problem for ordinary viscous liquid. Let the origin be taken at the sink point. Let the x-axis be along the wall of the convergent channel and let y-axis be perpendicular to the wall. Then we have the boundary layer equation in the form

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{-1}{\rho} \frac{\partial p}{\partial x} + K \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right)^n \quad \ldots (1)$$

$$\frac{\partial p}{\partial y} = 0 \quad \ldots (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots (3)$$

With the boundary conditions

$$\left.\begin{array}{l} u = v = 0, \quad y = 0 \\ u = U(x); \quad y = \infty \end{array}\right\} \quad \ldots (4)$$

where K is the kinematic viscosity for the power law liquid, n is the power law index, U(x) is the potential flow velocity Velocity along the wall of a convergent channel.

We assume that

$$U \frac{dU}{dx} = \frac{-1}{\rho} \frac{\partial p}{\partial x}$$

By using this equation (1) becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + K \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right)^n \qquad \ldots (5)$$

with boundary conditions

$$\left.\begin{array}{ll} u = V = 0; & y = 0 \\ u = U(x); & y \to \infty \end{array}\right) \qquad \ldots (6)$$

We introduce the stream function ψ as follows which satisfies the continuity equation

$$u = \frac{\partial \psi}{\partial y} = \psi_y, \quad v = - \frac{\partial \psi}{\partial x} = - \psi_x \qquad \ldots (7)$$

from equation (7), equation (5) becomes

$$\psi_y \psi_{yx} - \psi_x \psi_{yy} = U \frac{dU}{dx} + K \frac{\partial}{\partial y} (\psi_{yy})^n \; --- (8)$$

with the boundary conditions

$$\left.\begin{array}{ll} \psi_y = \psi_x = 0; & y = 0 \\ \psi_y = U(x); & y = \infty \end{array}\right) \qquad (9)$$

The potential flow velocity along the wall of convergent channel is given by

$$U(x) = - \frac{c}{x} \qquad (c > 0)$$

where x is measured along the wall of the channel and C is the strength of the sink at origin.

Hence for the application of similarity transformation we use the following dimensionless variables.

$$f(\eta) = \frac{\psi(x,y)}{\left(\frac{n+1}{2n}\right)^{\frac{1}{n+1}} C^{\frac{(2n-1)}{(n+1)}} K^{\frac{1}{(n+1)}} x^{\frac{(-2n+2)}{(n+1)}}} \quad \ldots \quad \ldots (10)$$

$$\eta = \frac{y.\left(\frac{2n}{n+1}\right)^{\frac{1}{(n+1)}} C^{\left(\frac{2-n}{n+1}\right)} x^{\frac{(n-3)}{(n+1)}}}{K^{1/n+1}} \quad \ldots (11)$$

from (10) and (11) we obtained

$$u = -Cf' x^{-1}$$

$$u\frac{\partial u}{\partial x} = \psi_y \psi_{yx}$$

$$= -C^2 f'^2 x^{-3} - \left(\frac{2n}{n+1}\right)^{\frac{1}{n+1}} \left(\frac{n-3}{n+1}\right) C^{\frac{(n+4)}{(n+1)}} . \frac{y f' f''}{K^{1/(n+1)}} \quad \ldots (12)$$

$$v = -\frac{\partial \psi}{\partial x} = -\psi_x$$

$$= \left(\frac{n+1}{2n}\right)^{1/n+1} K^{1/n+1} f.\left(\frac{2-2n}{n+1}\right) . x^{\frac{(-3n+1)}{(n+1)}} C^{\frac{(2n-1)}{(n+1)}} +$$

$$+ \; c \; \left(\frac{n-3}{n+1} \right) \; y \, f' \, x^{\frac{(2n-2)}{(n+1)}}$$

$$v \frac{\partial u}{\partial y} = -\psi_x \psi_{yy}$$

$$= \left(\frac{2-2n}{n+1} \right) c^2 \, f \, f'' \, x^{-3}$$

$$+ \frac{\left(\frac{n-3}{n+1} \right) \left(\frac{2n}{n+1} \right)^{1/(n+1)} c^{\frac{(n+4)}{(n+1)}} \, \eta f' f'' \, x^{\frac{(-2n-6)}{(n+1)}}}{K^{1/(n+1)}} \quad \ldots \qquad \ldots(13)$$

$$U \frac{du}{dx} = -c^2 \, x^{-3} \qquad \ldots(14)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^n = \frac{\partial}{\partial y}\left(\psi_{yy}\right)^n$$

$$= \frac{n.\left(\frac{2n}{n+1} \right) .c^2 . \, x^{-3} \, f''' \, (f'')^{n-1}}{K} \qquad \ldots(15)$$

Due to the equations (12), (13), (14) and (15), equation (8) takes the form

$$n. \left(\frac{2n}{n+1} \right) f''' \, (f'')^{n-1} + \frac{2-2n}{n+1} \, ff'' + 1 - f'^2 = 0 \qquad \ldots(16)$$

subject to the boundary conditions.

$$\left.\begin{aligned} f' = f = 0; \quad & \eta = 0 \\ f' \longrightarrow 1; \quad & \eta \longrightarrow \infty \end{aligned}\right) \qquad \ldots(17)$$

Equation (16) is the generalized equation for the boundary layer flow of a power law liquid past along the wall of the convergent channel. In particular if $n = 1$, then equation (16) will be reduced to the Falkner-Skan equation for convergent channel.

For the solution of equation (16) we can use Nachtsheim-Swigert iterative scheme and fourth order Range Kutta method. We can also use the FORTRAN programme for this problem which can be run over Burroughs 6700 computer for different values of the power law index n. Thus we can determine the local non-dimensional skin friction coefficient, displacement thickness, momentum thickness and kinetic energy thickness. We would like to investigate these in our further research work.

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