CHAPTER-I

Brief Survey of the developments in the solution of Navier-Stokes Equations

Introduction :

In this Chapter in Section 1 we gave the brief survey of the viscous flow theory. In Section 2 we gave some exact solution and Major developments in exact solutions of Navier-Stokes equations. Section 3 consists of some approximate solutions and Major developments in approximate solutions of the Navier-Stokes Equations.

1. Fundamental Equations of the flow of viscous fluids

(a) Equation of state :

$$F(P, \rho, T) = 0 \qquad \ldots (1.1)$$

where P : Pressure,

ρ : density,

T : Temperature.

For perfect gas the equation of state is given by

$$P = \rho R T \qquad \ldots (1.2)$$

where R is the gas constant.

(b) Equation of Continuity (Conservation of Mass)

The equation of continuity for compressible viscous fluid in cartesion tensor form is given by

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j} (\rho V_i) = 0 \qquad \ldots (1.3)$$

where ρ : the density of the fluid.

V_i : The components of fluid velocity.

In vector form the equation of continuity can be written as

$$\frac{\partial \rho}{\partial t} + \operatorname{div} (\rho \bar{V}) = 0 \qquad \ldots (1.4)$$

for incompressible fluid it is reduced to the form

$$\operatorname{div} \bar{V} = 0 \qquad \ldots (1.5)$$

(c) Equations of Motion (Navier-Stokes' Equations)

The equations of linear momentum for a viscous compressible fluid can be written in tensor form.

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho f_i + \frac{\partial T_{ij}}{\partial x_j} \qquad \ldots (1.6)$$

($i = 1$ to 3, $j = 1$ to 3)

where

v_i : represent the velocity of the fluid element.

f_i : the body force per unit mass.

ρ : the density of the fluid.

T_{ij} : The stress per unit area in the x_i direction. On an element of the surface where outward normal is in the x_j direction. Thus T_{11}, T_{22} and T_{33} represent tensions, while the remaining six stresses T_{12}, T_{13}, T_{21}, T_{23}, T_{31} and T_{32} are shear stresses acting through the agency of viscosity or internal friction in the fluid. It

can be easily shown that for these shear stresses or viscous stresses $T_{ij} = T_{ji}$
i.e. T_{ij} is a symmetric tensor.

To obtain the velocity distribution we use the following expressions, for the viscous stresses in terms of velocity gradient and fluid property.

For Newtonian fluid the constitutive equation

$$T_{ij} = 2\mu\, e_{ij} - 2/3\, \mu\, e_{kk}\, \delta_{ij} \qquad \ldots (1.7)$$

where

e_{ij} : pure strain given by

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \qquad \ldots (1.8)$$

and 'μ' is the constant coefficient of viscosity.
Substituting these in equation (1) we get

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j} \quad \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right) \qquad \ldots (1.9)$$

These are the three Navier-Stoke's Equations of motion of a viscous compressible fluid. For an incompressible fluid, it can be reduced to the vector form

$$\rho \frac{D\bar{v}}{Dt} = \rho \bar{f} - \nabla P + \mu \nabla^2 \bar{v} \qquad \text{... (1.10)}$$

where $D/Dt = \partial / \partial t + (\bar{\nabla} \cdot V)$.

is known as material derivative.

(d) The Navier-Stokes equations of Motion for gases:

The viscous stress component must be linear function, of the rate of strain component, if $\Delta = e_{kk}$ is the dilation rate, it can be shown that the non-dilational part, $e_{ij} - \frac{1}{3}\Delta\delta_{ij}$, of the rate of strain of pure shearing motions which must be resisted by viscous shear stress making a contribution $2\mu \left(e_{ij} - \frac{1}{3} \Delta \delta_{ij} \right)$ to the stress tensor. Also the isotropic part of the stress tensor, $\left(\frac{1}{3} P_{kk} \right) \delta_{ij}$, may be a function of the dilation rate, however, for a monatonic gas, in which the internal energy is purely translational $\left(- \frac{1}{2} P_{kk} \right)$ is the energy per unit volume and so $\left(- \frac{1}{3} P_{kk} \right)$ is the thermodynamics pressure $P = 2/3 \rho E$. Where E is the internal energy per unit mass. Although the thermodynamical identity

$$P = (\gamma - 1) \rho E$$

$$(\gamma = 5/3 \text{ for a monatomic gas})$$

is not expected to hold except in equilibrium conditions, it is convenient in fluid dynamics to Re-define pressure so that it does; it is only in work on relaxation phenomena that the thermodynamic definition of pressure becomes unsuitable so for monatomic gases the stress-rate of strain relationship is

$$P_{ij} = - P\delta_{ij} + 2\mu \ (e_{ij} - 1/3 \ \Delta\delta_{ij})$$

for a gas with diatomic molecules possessing rotational energy as well as translational energy this relationship has to be modified since for rotational energy lags behind the translational energy, i.e. the adjustment to a new state of the rotational energy takes a few more molecular collisions than for the translational energy, for such a gas under a positive dilation rate Δ, the temperature is falling and since the rotational energy is lagging slightly behind the falling translational energy, the thermodynamics pressure $P = (\gamma-1) \rho E$. remains slightly greater than $(-1/3 \ P_{kk})$ by an amount $\beta\Delta$, say, where β is known as bulk viscosity and so the stress-rate of strain relationship becomes

$$P_{ij} = (\beta\Delta - p) \ \delta_{ij} + 2\mu \ (e_{ij} - 1/3 \ \Delta\delta_{ij}) \qquad \ldots (1.11)$$

When the presence of energy in other forms is important, the time lag is reaching equilibrium is too great for a simple linear dependence of the stresses on the first derivative of the velocity to be generally valid, and the Relaxation process then have to be considered explicitly, the importance of the bulk viscosity concept is that when available, it enables equilibrium thermodynamics to be used; like the viscosity, the bulk viscosity varies with temperature, it may be noted that since the bulk viscosity is only important when Δ is sufficiently large its importance in practice has been restricted to such a matters as sound absorption and acoustic streaming and shock waves.

An alternative approach to the stress-rate of strain relation may be made through the kinetic theory of gases, the kinetic theory gives more than indicated above as it also determines the viscosity coefficient in terms in molecular properties.

Substituting then for the stress tensor P_{ij} in the equation of motion (-) and simplifying we have

$$\rho \quad \frac{\partial v}{\partial t} + \nabla \left(\frac{1}{2} v^2 \right) + WV$$

$$= \rho E - \nabla P - V \wedge \nabla \wedge (\mu V) - V \quad \nabla^2 \mu$$

$$+ \nabla \mu \wedge W + \nabla (\beta \Delta) + 4/3 \quad \nabla (\mu \Delta)$$

$$+ V (V \nabla \mu) - \Delta \nabla \mu . \qquad \ldots (1.12)$$

where $W = V \wedge V$

the three scaler equations embodied in this vector equation are usually referred to as Navier-Stokes' equations.

The original derivative by Navier applied only to incompressible flow, stokes considered compressible flow in its compressible form with o and μ constant and $\Delta = 0$, the above equation reduces to

$$\frac{\partial v}{\partial t} + W \wedge V = f - V \left(\frac{p}{\rho} + \frac{1}{2} v^2 \right) - \nu \, \nabla \wedge W \qquad \ldots (1.13)$$

It should be noted that in the incompressible case the pressure is not a thermodynamics veriable and is defined as (mean of normal stresses) being equal to the hydrostatic pressure, when there is no motion.

The form of the stress-rate of strain law, based on it, is on a linear relationship might be expected to be of limited applicability; however, Judged indirectly by comparison of its consequence with experiment the range of validity is very wide and it is only with fluids of complex molecular structure at high rates of strain that the linear relationship has been found inadequate.

e) Equation of Energy :

The equation of energy for viscous incompressible fluid in cartesion coordinates.

$$\rho C_v \frac{DT}{Dt} = \frac{\partial Q}{\partial t} + K \nabla^2 T + \phi \qquad \ldots (1.4)$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\phi = 2\mu \left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z^2}\right)^2 \right.$$

$$\left. + \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right.$$

$$D/Dt = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}$$

ρ : density

T : Temperature

C_v : specific heat

K : Thermal conductivity

ϕ : dissipation function

Q : External heat.

2. Some exact solutions of the Navier stokes equation and major developments.

(a) Some Exact Solution of Navier-Stokes Equation

There is no general method for the solution of Navier-Stokes Equation of motion. The main reason is that these equations are nonlinear therefore, at the present time, there is no known general method for solving these nonlinear differential equations. Only for a small number of special cases we can find the exact solution of these equations.

In obtaining these exact solutions of the Navier-Stokes Equations the following assumption have been made:

i) Simple configuration for the flow pattern is considered.

ii) the fluid is incompressible i.e. the density of the fluid is assumed to be contant.

iii) the coefficient of viscosity, and that of heat conduction are constant. Under these assumption we can solve for the three components of velocity in space and the pressure of the fluid from the equation of motion and the equation of continuity (Navier-Stokes equations of a viscous fluid in the ordinary sense). After the velocity and pressure distributions in the flow field are known, the temperature distribution will be determined by energy equation.

Although the Navier-Stokes equations of a viscous incompressible fluid are much simpler than those equations for a viscous compressible fluid, and hence not solvable in general. Only in some simple configurations the nonlinear terms of the Navier-Stokes equations drop out or become simple enough, then the equation are solvable. In most of these cases only one component of the velocity is different from zero and the independent variables are limited to two namely, either two spatial coordinates or one spatial coordinate and the time, we are going to discuss some of these cases.

The known exact solutions for the temperature distribution in the incompressible fluid cases are fewer than those for the velocity distribution. Whenever the expression for the velocity distributions become too complicated, the exact solution of the energy equation is difficult to solve. Another current practice in finding the temperature distribution approximately is to drop the viscous dissipation terms in the energy equations and then to solve for the temperature distribution.

For a viscous compressible fluid, the only possible exact solution seems to be that with one spatial coordinates one dimensional steady flow, which is the problem describing the shock wave thickness.

For the calculation of practical importance, the fundamental equation of a viscous compressible fluid have to be further simplified by other assumption so that more complicated configuration may be treated approximately.

(b) Major Developments in the exact solutions of Navier-Stokes equation :

Poiseulle, J [18] studied flow in a circular pipe and obtained velocity distribution and temperature distribution and calculated volume rate of flow and coefficient of skin friction.

Stokes, G.G. [20] studied unsteady incompressible flow with constant fluid properties.

Couette, [4] studied parallel flow between two parallel plates and obtained the velocity distribution and temperature distribution and calculated volume rate of flow and coefficient of skin friction.

Poiseulle, J [18] studied the incompressible flow in tubes of uniform cross section and obtained volume rate of flow in annular, Elliptic, Equilateral, triangular and Rectangular cross section.

Jeffery and Hamel, G. [8 & 14] studied the flow in a convergent and divergent channel. Thus the solution of this problem may be expressed in terms of elliptical integrals, the different possible cases of this type have been discussed by Hamel.

Couette [4] Investigated flow between two concentric rotating cylinders and obtained velocity distribution and temperature distribution.

Prandtl, L. [19] studied the Relation between viscosity and temperature, this result had been varyfied by experiment.

Syzmanski, F. [21] studied the starting flow in a pipe with constant pressure gradient and obtained velocity distribution for unsteady flow in a circular pipe.

Illingworth, C.R. [11] tried to find some exact solution of the Navier-Stokes equations of a viscous compressible fluid and found that only solutionssimilar to coutle flow of an incompressible fluid could be obtained in simple closed form and that no simple solution corresponding to poiseuilla flow or other exact solutions of an incompressible fluid be found. He obtained exact solutions for

i) plane couetle flow past a porous flat plate.

ii) simple shearing motion between rotating cylinders and

iii) circulating flow round a circular cylinder with suction at the surface.

Nanda, R.S. [17] studied two dimensional steady laminar flow of viscous incompressible fluid between two parallel plates and obtained velocity distribution and temperature distribution for plane couetle flow with transpiration cooling.

Bansal, J. [2] studied Generalised plane couette flow with uniform suction and injection at the stationary plate and plane couetle flow with transpiration cooling.

Jain, N.C. [13] studied couetle flow with transpiration cooling when the viscosity of the fluid depends on temperature and studied with variable viscosity plane couette flow and obtained the exact solution.

Heusenblas, H. [9] studied variable viscosity plane poiseulla flow and obtained velocity distribution and temperature distribution.

Sudhanshu Kumar Ghoshal [2] obtained non-regular solution of the Navier-Stokes equations for an incompressible three dimensional flow.

Alan, R. Elcrat [1] studied a boundary value problem which hielded exact solution of the Navier-Stokes Equations, for the flow between two infinite co-axial and permeable discs, and desired some conclusions.

Dorrepaal, J.M. [6] obtained similarity solution for the flow impinging on a flat wall of arbitrary angle of incidance the technique which he has adopted is similar to a method used by Jeffery and Pezegrine 1

3. Approximate Solutions and Major development in Approximate Solutions of Navier-Stokes Equations.

(a) Approximate solutions of Navier-Stokes Equations :

In this section we shall discuss the approximate solutions of the Navier-Stokes equations for problems of the very small Reynolds number.

Very few exact solution of the Navier-Stokes equations are available and none of these deal with flow past a finite body, because of these limitation approximate equation have been derived from the Navier-Stokes equations, these approximate solution fall into two categories :

i) The equations of stokes and oseen for flows at low Reynolds number and

ii) Prandtl boundary layer equations for flow at high Reynolds number.

In these problems the frictional forces are much larger than the inertial forces the approximate solution are supposed to hold true for systems with Reynolds number below 1, these problem may be divided into two groups; one group deals with the motion of very small bodies with very small speeds such as the falling of sand segments in water or the falling of just in air. Let us consider the case of a small sphere falling in air. If the Reynolds number $Re = Vd / \nu$ is equal to 1, the diameter 'd' of the sphere will be only 1 mm and its velocity 'V' 1.4 cm/sec for an average value ν for air which is approximately ν = 0.14 cm^2/sec it is indeed a very small body with small velocity. The other group deals with the flow of liquids of large viscosity such as the motion of oil in the theory of lubrication. In both groups the velocity of fluid is always small.

One of the simplest cases of a tiny body moving slowly in the viscous fluid is the passage of a sphere in a viscous fluid. The solution of this problem was given by Stokes the solutions for other simple cases such as a circular cylinder, ellipsoids, and the like, have also been worked out. They are given in lamb's classical book of hydrodynamics.

Eventhough the stokes solution gives good results for the drag of sphere of Re $<$ 1, the flow pattern at large distances from the sphere given by stokes solution is not correct, oseen improved stokes solution so that a correct overall picture of the flow field is obtained.

(b) Major Development in Approximate solution of Navier-Stokes' Equations.

Verzbinskaja Ju, S. [23] investigated a new coordinate system in the galerkin method for the solving the Navier-Stokes equations. This system is complete in L 2 (Ω) metric in the space of solenoid vectors.

Some numerical results were obtained by using computer for turbulent flow.

Ivonov, K.P. and Kapiska, A. [12] studied an approximate solution of the Navier-Stokes equation for an arbitrary domain with a smooth boundary, using the variational difference scheme technique.

Dey, S.K. [5] studied most numerical solutions of the Navier-Stokes equation have been obtained by explit finite difference method.

Ma, Yan Wen [16] studied two difference schemes are proposed for reducing the oscillations near a shock in solving the Navier-Stokes equations one is a scheme obtained by using a fine increment parameter, the other is a one step difference scheme with an adjustable parameter.

Belov, Yu. Ya. [3] studied approximations of the Navier-Stokes equations in which the equation of motion are third order equations nonlocal with respect to time, and containing a parameter $\epsilon > 0$, for the case of first boundary value problem. They proved the corresponding problem for the Navier-Stokes equations.

Foias, C., Manley, O.P., Temam, R., Treve, Y.M. [7] studied recent efforts to estimate the number of modes sufficient for the approximate solution of Navier-Stokes Equation in two dimensional and three dimensional motions.

Kheshgi, Haroon, Luskin and Mitchell [15] studied a numerical analysis of incompressible viscous fluid using the variable sign penalty method.

Hamza, E.A.; MacDonald, D.A. [10] studied viscous incompressible fluid is contained between two parallel disks, at time 't' are spaced a distance $H / \sqrt{1 - \alpha t}$ apart and are rotating with angular velocities proportional to Ω $(1 - \alpha t)^{-1}$ the governing Navier-Stokes equation was reduced to a set of ordinary differential equation they obtained the approximate solution to these equations.

Von Karman's [24] studied flow due to rotating disc and obtained approximate solution of Navier-Stokes equation in cylindrical polar coordinate in absence of body force by an approximate method which was later improved by Cochran, W.G. and other workers.

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