

Unit 6

Distributions of Certain Stopping Times Based on Bernoulli Random Variables

Abstract

Based on a sequence of independent Bernoulli trials, certain stopping rules are defined. Such rules can be used in Statistical process control techniques. Based on a sequence of Bernoulli trial a bivariate model has been introduced.

1. Introduction

In this article, we define certain stopping rules based on a sequence of i.i.d. Bernoulli trials and find the distribution of the corresponding stopping random variable. In industrial applications, a success may refer to the status of confirming the quality or in medical follow-up, it may refer to survival or recovery from a disease. In process control techniques, such stopping r.v. may correspond to time to stop (get signal) and declare that the process has gone out of control.

Further, Average Time to Signal (ATS) being an important characteristic of the process control technique, ATS plays an important role. Hence, for the considered stopping rules we compute the ATS.

More specifically, Groups Runs (GR) control charts based on the Rule 4 defined in sub-section 2.4, performs very well compared to many such charts. As such to reconfirm the result, in this example, expected value has been obtained by different methods.

A discrete bivariate model is developed and its p.g.f. is derived. A Markov Chain representation is also given.

2. Stopping Rules

We consider four stopping rules based on a sequence of independent Bernoulli trials. Based on the recurrence relation and the initial conditions, the distributions of the times to stop for the considered rules have been obtained.

2.1 Particular Stopping Rules

Consider a sequence of independent trials with p , the probability of getting Head (H). In the following we refer Head, Tail respectively as success and failure.

Rule 1. Consider the stopping rule 'stop after getting the first success'. The stopping r.v. N has waiting time distribution with

$$P\{N = k\} = q^{k-1} p, \quad k = 1, 2, \dots \quad (1)$$

This can be obtained by using the recurrence relation. Note that $P_k = P\{N = k\}$ is given by $P_1 = p$, and $P_n = (1 - P_1 - \dots - P_{n-1}) p$, for $n \geq 2$.

Thus, we have the recurrence relation

$$\begin{aligned} P_n &= P_{n-1} - pP_{n-1}, \text{ for } n \geq 3 \\ &= q P_{n-1} \\ &= q q P_{n-2} = q^{n-1} P_1 = q^{n-1} p. \end{aligned} \quad (2)$$

Hence (1) follows.

Rule 2. Consider the stopping rule 'stop after TH is obtained for the first time'.

The distribution of N is given by

$$P_1 = 0, P_2 = pq, P_3 = pq, P_4 = (1 - pq)pq, \text{ and}$$

$$P_n = (1 - P_1 - \dots - P_{n-2}) p, \text{ for } n \geq 2.$$

Further, by using combinatorial arguments P_n can be obtained as follows.

For $n \geq 2$ the event $\{N = n\}$ occurs if and only if a sequence of k heads is followed by a sequence of $n-2-k$ tails and followed by a tail and a head.

Hence,

$$P_n = \sum_{k=0}^{n-2} p^k (pq) q^{n-2-k}$$

$$= \begin{cases} pq \left(\frac{q^{n-1} - p^{n-1}}{q - p} \right) & \text{if } p \neq q \\ \frac{n-1}{2^n} & \text{if } p = q = 1/2. \end{cases} \quad (3)$$

Moreover, we have $P_1 = 0$, $P_2 = qp$, $P_3 = pq$, $P_4 = (1 - qp)pq$ and

$$\begin{aligned} P_n &= (1 - P_2 - \dots - P_{n-2})pq, \text{ for } n \geq 4. \\ P_{n-1} &= (1 - P_2 - \dots - P_{n-3})pq, \text{ for } n \geq 5. \end{aligned} \quad (4)$$

Hence, from (4) we have the recurrence relation,

$$P_n = P_{n-1} - pq P_{n-2}, \text{ for } n \geq 5.$$

In the following we shall show that $P_n = \frac{pq}{q-p} (q^{n-1} - p^{n-1})$ is the solution

of (4). Let $p \neq q$.

$$\begin{aligned} \text{Consider } P_1 + \dots + P_{n-2} &= \sum_{k=1}^{n-2} \frac{pq}{q-p} (q^{k-1} - p^{k-1}) \\ &= \frac{pq}{q-p} \left\{ \frac{1 - q^{n-2}}{1 - q} - \frac{1 - p^{n-2}}{1 - p} \right\} \\ &= \frac{1}{q-p} \{q - q^{n-1} - p + p^{n-1}\}. \end{aligned}$$

Hence,

$$\begin{aligned} pq \{1 - (P_1 + \dots + P_{n-2})\} &= \frac{pq}{q-p} \{q - p - q + q^{n-1} + p - p^{n-1}\} \\ &= \frac{pq^2}{q-p} \{q^{n-1} - p^{n-1}\} \\ &= P_n \end{aligned}$$

Similarly, the result can be proved when $p = q$. It is easy to observe that P_n 's are

positive. In the following, we verify that $\sum_2^\infty p_n = 1$. Consider, (for $p \neq q$),

$$\begin{aligned} \sum_{n=2}^\infty p_n &= \frac{pq}{p-q} \left\{ \sum_{n=2}^\infty q^{n-1} - \sum_{n=2}^\infty p^{n-1} \right\} \\ &= \frac{pq}{p-q} \left\{ \frac{q}{1-q} - \frac{p}{1-p} \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{pq}{p-q} \left\{ \frac{q}{p} - \frac{p}{q} \right\} \\
&= \frac{q^2 - p^2}{p-q} \\
&= p+q=1.
\end{aligned}$$

If $p = q = 1/2$, $P_n = \frac{n-1}{2^n}$ and

$$\sum_{n=2}^{\infty} P_n = \sum_{n=2}^{\infty} (n-1) \frac{1}{2^n} = \frac{1}{4} \sum_{n=2}^{\infty} (n-1) \left(\frac{1}{2}\right)^{n-2} = \frac{1}{4} \left(1 - \frac{1}{2}\right)^{-2} = 1.$$

Rule 3. Consider the rule 'stop as soon as 2 consecutive successes are obtained'.

In this case,

$$P_1 = 0, P_2 = p^2, P_3 = qp^2, P_4 = qp^2 \text{ and } P_5 = (1-p^2)p^2q \text{ and}$$

$$P_n = pq^2 (1 - P_1 - \dots - P_{n-3}), \text{ for } n \geq 4.$$

$$P_{n-1} = qp^2 (1 - P_1 - \dots - P_{n-4}), \text{ for } n \geq 5.$$

Thus, we get the difference equation.

$$P_n = P_{n-1} - qp^2 P_{n-3}, \text{ for } n \geq 5. \quad (5)$$

Alternatively, an expression for P_n can be obtained by using combinatorial arguments as follows

$$P_n = p^2 q \sum_{r=0}^n \binom{n-r-2}{r} p^r q^{n-r-4} \quad (6)$$

In the following, we verify that

$$\sum_{n=2}^{\infty} P_n = 1$$

Consider,

$$\begin{aligned}
\sum_{n=4}^{\infty} P_n &= p^2 q \sum_{n=4}^{\infty} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-r-2}{r} p^r q^{n-r-4} \\
&= p^2 q \left\{ \sum_{n=4}^{\infty} \binom{n-2}{0} q^{n-3} + \sum_{n=4}^{\infty} \sum_{r=1}^n \binom{n-r-2}{r} p^r q^{n-3-r} \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{r=1}^{\infty} \sum_{n \geq 2r+2}^{\infty} \binom{n-r-2}{r} p^r q^{n-3-r} \right\} \quad \text{put } k = n - r - 3
\end{aligned}$$

$$\begin{aligned}
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{r=1}^{\infty} p^r \sum_{k=r-1}^{\infty} \binom{k+1}{r} q^k \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{r=1}^{\infty} p^r \sum_{m=r}^{\infty} \binom{m}{r} q^{m-1} \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{r=1}^{\infty} p^r \sum_{m=r}^{\infty} \binom{m}{r} q^{m-1} \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{m=1}^{\infty} \sum_{r=1}^m p^r \binom{m}{r} q^{m-1} \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{m=1}^{\infty} q^{m-1} \sum_{r=1}^m \binom{m}{r} p^r \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{m=1}^{\infty} q^{m-1} [(1+p)^m - 1] \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \sum_{m=1}^{\infty} (1-p^2)^m \frac{1}{q} - \sum_{m=1}^{\infty} q^{m-1} \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \frac{1-p^2}{p^2 q} - \frac{1}{p} \right\} \\
&= p^2 q \left\{ \frac{q}{1-q} + \frac{1+p}{p^2} - \frac{1}{p} \right\} \\
&= p^2 q \left\{ \frac{1+p-p^2}{p^2} \right\} \\
&= q \{1 + p - p^2\}
\end{aligned}$$

$$P_2 + P_3 = p^2 + p^2 q$$

$$\text{Hence, } \sum_{n=2}^{\infty} P_n = P_2 + P_3 + \sum_{n=4}^{\infty} P_n$$

$$= p^2 + p^2 q + (q + pq - p^2 q)$$

$$= p^2 + (1-p)(1+p)$$

$$= 1.$$

The generating function $G(s)$

The generating function $G(s)$ is given by

$$G(s) = s^2 p^2 + s^3 p^2 q + \sum_{n=4}^{\infty} s^n P_n$$

$$\begin{aligned}
&= s^2 p^2 + s^3 p^2 q + \sum_{n=4}^{\infty} s^n \{P_{n-1} - qp^2 P_{n-3}\} \\
&= s^2 p^2 + s^3 p^2 q + s \sum_{k=3}^{\infty} s^k P_k - s^3 \sum_{k=1}^{\infty} s^k P_k qp^2 \\
&= s^2 p^2 + s^3 p^2 q + s [G(s) - s^2 p^2] - s^3 qp^2 G(s) \\
G(s) &= \frac{s^2 p^2 + s^3 qp^2 - s^3 p^2}{1 - s + s^3 qp^2} \\
&= \frac{s^2 p^2 - s^3 p^3}{1 - s + s^3 qp^2} \\
G(1) &= \frac{p^2 q}{p^2 q} = 1 \\
G'(s) &= \frac{2sp^2 - 3s^2 p^3}{(1 - s + s^3 qp^2)^2} - \frac{s^2 p^2 - s^3 p^3}{(1 - s + s^3 qp^2)^2} \{-1 + 3s^2 p^2 q\} \\
G'(0) &= 0 \\
G'(1) &= \frac{2-3p}{q} - \frac{(-1+3p^2 q)}{qp^2} \\
&= \frac{2p^2 - 3p^3 + 1 - 3p^2 q}{qp^2} \\
&= \frac{1 + p^2(2-3q) - 3p^3}{qp^2} \\
&= \frac{1 + p^2(1-q+1+q-q) - 3p^3}{qp^2} \\
&= \frac{1 + 2p^3 - qp^2 - 3p^3}{qp^2} \\
&= \frac{1 - p^3 - p^2 q}{qp^2} \\
&= \frac{1 - p^2(p+q)}{qp^2}
\end{aligned}$$

$$\text{Hence } E(X) = G'(1) = \frac{1-p^2}{qp^2}$$

Rule 4. Consider the stopping rule 'stop if the first trial is success or by the time two successive trials are successes'. Here,

$$P_1 = p, P_2 = 0, P_3 = qp^2, P_4 = q^2p^2 = P_5, \dots$$

and $P_n = qp^2 \{1 - P_1 - \dots - P_{n-3}\}, \quad \text{for } n \geq 5.$

That is

$$P_n = P_{n-1} - qp^2 P_{n-3}. \quad (7)$$

By using combinatorial arguments it can be shown that

$$P_n = p^2 q^2 \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-r-3}{r} p^r q^{n-r-4}. \quad (8)$$

It can be verified that

$$\sum_{n=1}^{\infty} P_n = 1$$

The difference equation obtained in case of rule 3 and here are the same but, initial conditions are different.

It is to be noted that if ' P_n ' is a solution of (7) then P_{n+1} and cP_n are also solutions of (7), under no boundary conditions.

In the following we obtain expression for $E(N)$ by direct computation, using p.g.f and the conditioning technique.

(i) **Direct Method**

$$\text{Let } T_0 = p + 3(1-p)p^2 + 4(1-p)^2p^2 \quad (9)$$

$$\begin{aligned} \text{Then } E(N) &= \sum_1^{\infty} nP_n = T_0 + p^2(1-p)^2 \sum_{n=5}^{\infty} \left\{ n \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-3-r}{r} p^r (1-p)^{n-4-r} \right\} \\ &= T_0 + p^2(1-p)^2 \left\{ \sum_1^{\infty} (n^* + 4) \sum_{r=0}^{\lfloor \frac{n^*}{2} \rfloor + 1} \binom{n^* + 1 - r}{r} p^r (1-p)^{n^* - r} \right\} \\ &\quad \dots (\text{by putting } n^* = n - 4) \quad (10) \end{aligned}$$

The quantity inside the bracket is given by

$$\sum_{n^*=1}^{\infty} (n^* + 4)(1-p)^{n^*} + \sum_{r=1}^{\infty} p^r \sum_{n^*=r-1}^{\infty} \binom{n^* + 1}{r} (n^* + 4 + r)(1-p)^{n^*}$$

(by Putting $n' = n^* - r$)

$$\begin{aligned}
&= \sum_{n=1}^{\infty} (n+4)(1-p)^n + \sum_{r=1}^{\infty} p^r \sum_{n=r-1}^{\infty} \binom{n+1}{r} (n+4+r)(1-p^n) \\
&= T_1 + T_2 \quad (\text{Say}). \tag{11}
\end{aligned}$$

Consider

$$\begin{aligned}
T_2 &= \sum_{n=0}^{\infty} \sum_{r=1}^{n+1} \binom{n+1}{r} p^r (n+4+r)(1-p)^n \\
&= \sum_{n=0}^{\infty} (1-p)^n \sum_{r=1}^{n+1} \binom{n+1}{r} p^r (n+4+r) \\
&= \sum_{n=0}^{\infty} (1-p)^n h(n, p)
\end{aligned}$$

$$\begin{aligned}
\text{where } h(n, p) &= \sum_{r=1}^{n+1} \binom{n+1}{r} p^r (n+4+r) \\
&= (n+4) \sum_{r=1}^{n+1} \binom{n+1}{r} p^r + \sum_{r=1}^{n+1} r \binom{n+1}{r} p^r \\
&= (n+4) [(1+p)^{n+1} - 1] + (n+1)p(1+p)^n
\end{aligned}$$

Hence

$$\begin{aligned}
T_2 &= \sum_{n=0}^{\infty} (1-p)^n \{ (n+4) [(1+p)^{n+1} - 1] + (n+1)p(1+p)^n \} \\
&= \sum_{n=0}^{\infty} (1-p)^n \{ (1+p)^n [(1+p)(n+4) + (n+1)p] - (n+4) \} \\
&= \sum_{n=0}^{\infty} (1-p^2)^n (n+2np+5p+4) - \sum_{n=0}^{\infty} (n+4)(1-p)^n \\
&= \frac{(1-p^2)}{(p^2)} + \frac{2p(1-p^2)}{(p^2)^2} + \frac{5p}{p^2} + \frac{4}{p^2} - \sum_{n=0}^{\infty} (n+4)(1-p)^n \\
&= \frac{1}{p^4} \{ 1 + 2p + 3p^2 + 3p^3 \} - \sum_{n=0}^{\infty} (n+4)(1-p)^n \\
&= T_3 - T_4 \quad (\text{Say}) \tag{12}
\end{aligned}$$

Hence,

$$\begin{aligned}
T_1 - T_4 &= \sum_{n=1}^{\infty} (n+4)(1-p)^n - \sum_{n=0}^{\infty} (n+4)(1-p)^n \\
&= -4. \tag{13}
\end{aligned}$$

Hence, from (10), (12), (13) and (9) in sequence, we have

$$\begin{aligned}
E(N) &= T_0 + p^2(1-p)^2 \{ T_1 + T_2 \} \\
&= T_0 + p^2(1-p)^2 \{ T_1 + T_3 - T_4 \} \\
&= T_0 + p^2(1-p)^2 \{ T_3 - 4 \} \\
&= \{ p + 3(1-p)p^2 + 4p^2(1-p)^2 \} + p^2(1-p)^2 \left\{ \frac{1}{p^4} \{ 1 + 2p + 3p^2 + 3p^3 \} - 4 \right\} \\
&= \{ p + 3(1-p)p^2 + 4p^2(1-p)^2 \} + (1-p)^2 \left\{ \frac{1}{p^2} \{ 1 + 2p + 3p^2 + 3p^3 \} - 4p^2 \right\} \\
&= \{ p + 3(1-p)p^2 + (1-p)^2(1 + 2p + 3p^2 + 3p^3) / p^2 \} \\
&= \frac{p^3 + 3p^4 - 3p^5 + (1 - 2p + p^2) \{ 1 + 2p + 3p^2 + 3p^3 \}}{p^2} \\
&= \frac{1}{p^2}
\end{aligned}$$

(ii) $E(N)$ by using Generating function: -

$$\begin{aligned}
G(s) &= \sum_{n=1}^{\infty} s^n P_n \\
&= sp + s^3 p^2 q + \sum_{n=4}^{\infty} s^n P_n \\
&= sp + s^3 p^2 q + \sum_{n=4}^{\infty} s^n \{ P_{n-1} - qp^2 P_{n-3} \} \\
&= sp + s^3 p^2 q + s \sum_{k=3}^{\infty} s^k P_k - qp^2 s^3 \sum_{k=1}^{\infty} s^k P_k
\end{aligned}$$

$$\begin{aligned}
\text{That is, } G(s) &= sp + s^3 q p^2 + s \{ G(s) - sp \} - qp^2 s^3 G(s) \\
&= sp + s^3 q p^2 - s^2 p + G(s) \{ s - s^3 qp^2 \}
\end{aligned}$$

Hence,

$$G(s) = \frac{sp + s^3 qp^2 - s^2 p}{1 - s + s^3 qp^2} \quad (14)$$

We note that, $G(1) = \frac{p + qp^2 - p}{qp^2} = 1.$

Differentiating G w.r.t. s ,

$$G'(s) = \frac{p + 3p^2 s^2 q^2 ps}{1 - s + s^3 qp^2} - \frac{\{ -1 + 3s^2 qp^2 \} \{ sp + s^3 p^2 - s^2 p \}}{(1 - s + s^3 qp^2)^2}.$$

We verify that $P[X = 1] = G'(0) = p$. Further,

$$\begin{aligned} E[X] = G'(1) &= \frac{3qp^2 - q}{qp^2} - \frac{(3qp^2 - 1)qp^2}{qp^2} \\ &= \frac{3p^2q - p - 3q^2p^4 - p^2q + 1}{p^2q} \\ &= \frac{1}{p^2}. \end{aligned}$$

(iii) *E(N) by Using Conditional Distribution*

Further if, X_3 and X_4 are the stopping r.v.'s correspond to the rules defined in 3 and 4 respectively then, we note that

$$P[X_4 = 1] = p \text{ and}$$

$$P[X_4 = k + 1 \mid X_4 > 1] = P[X_3 = k] \quad (16)$$

$$EX_4 = E[X_4 \mid X_4 \leq 1] P[X_4 \leq 1] + E[X_4 \mid X_4 > 1] P[X_4 > 1]$$

$$= p + q (EX_3 + 1) = p + q \left[\frac{1 - p^2}{qp^2} + 1 \right]$$

$$= p + \left[\frac{1 - p^2 + qp^2}{p^2} \right]$$

$$= p + \frac{1}{p^2} - 1 + q$$

$$= \frac{1}{p^2}.$$

3. *A Discrete Bivariate Model :*

Consider an experiment involving tossing of two coins I and II. The experiment and the stopping rule are as described in the following steps :

Step 1. Start tossing of coin I. If it results in tail stop the experiment else move to step 2.

Step 2. Start tossing coin II and move to step 1 after a tail has occurred.

Let N_1 and N_2 be the number of tosses of coin I and II respectively, by the time the experiment has stopped. Let N be the total number of tosses. Then $N = N_1 + N_2$ is the stopping random variable corresponding to the defined experiment. Let X_i (Y_i) is 1 if the i^{th} toss of the coin I (II) results in tail. We note

that the experiment is stopped as soon as $X_n = 1$ for the first time. Let $Z_j = X_1$ (or Y_m) if the j^{th} toss in the experiment corresponds to 1^{th} toss of the coin I (m^{th} toss of the coin II). The event $\{N_1 = r, N_2 = s\}$ occurs if and only if $\{X_1 = X_2 = \dots = X_{(r-1)} = 0 \text{ and } X_r = 1 \text{ and on the coin II } (r-1)^{\text{th}} \text{ tail is obtained on the } s^{\text{th}} \text{ toss of the coin II}\}$.

Hence $P\{N_1 = 1, N_2 = 0\} = P\{X_1 = 1\} = \alpha$ and $P\{N_1 = 2, N_2 = 1\} = P\{X_1 = 0, Y_1 = 1, X_2 = 1\} = (1 - \alpha)\beta\alpha$.

$$P\{N_1 = r, N_2 = s\}$$

$= P\{X_1 = X_2 = \dots = X_{(r-1)} = 0 \text{ and } X_r = 1 \text{ and on the coin II } (r-1)^{\text{th}} \text{ tail is obtained on the } s^{\text{th}} \text{ toss of the coin II}\}$.

$= P\{X_1 = X_2 = \dots = X_{(r-1)} = 0 \text{ and } X_r = 1\} P\{\text{on the coin II } (r-1)^{\text{th}} \text{ tail is obtained on the } s^{\text{th}} \text{ toss of the coin II}\}$.

$$= (1 - \alpha)^{r-1} \alpha \binom{s-1}{r-2} \beta^{r-1} (1 - \beta)^{s-r+1}. \quad (\text{Since for each } r, s \text{ the random variables}$$

$X_1, X_2, \dots, X_r$ and $Y_1, Y_2, \dots, Y_s$ are independent).

The joint probability mass functions (p.m.f.) of N_1 and N_2 is given by,

$$P(N_1 = r, N_2 = s) = \begin{cases} \alpha & \text{if } r = 1, s = 0 \\ (1 - \alpha)\beta\alpha & \text{if } r = 2, s = 1 \\ \binom{s-1}{r-2} [b(1 - B)]^{(r-1)} (1 - b) B^{s-r+1} & \text{if } r = 2, 3, \dots, (s + 1); \\ & s = 2, 3, \dots \end{cases}$$

3.1 The generating function of the random vector (N_1, N_2) :

$$\begin{aligned} P(s_1, s_2) &= E(s_1^{N_1} (\phi_Z(s_2))^{N_1-1}) \\ &= \frac{1}{\phi_Z(s_2)} E[(s_1 \phi_Z(s_2))^{N_1}] \\ &= \frac{1 - s_2(1 - \beta)}{s_2 \beta} \phi_{N_1}[s_1 \phi_Z(s_2)] \end{aligned}$$

$$\begin{aligned}
&= \frac{1-s_2(1-\beta)}{s_2\beta} \phi_{N_1} \left(s_1 \frac{s_2\beta}{1-(1-\beta)s_2} \right) \\
&= \frac{1-s_2(1-\beta)}{s_2\beta} \left(\frac{\frac{s_1s_2\beta\alpha}{1-(1-\beta)s_2}}{1-\frac{s_1s_2\beta(1-\alpha)}{1-(1-\beta)s_2}} \right) \\
&= s_1\alpha \left[\frac{1-s_2(1-\beta)}{1-(1-\beta)s_2-s_1s_2\beta(1-\alpha)} \right]
\end{aligned}$$

Generating function of N_1 is given by $P(s_1, 1) = \frac{s_1\alpha}{1-s_1(1-\alpha)}$.

Generating function of N_2 is given by $P(1, s_2) = \alpha \left[\frac{1-s_2(1-\beta)}{1-(1-\beta)s_2-s_2\beta(1-\alpha)} \right]$

3.2 A Markov Chain Representation :

Consider the process $\{Z_n\}$ corresponding to the above sequence of tossing of coins. The process $\{Z_n\}$ is a 4-state Markov Chain with states o, 1, O, I respectively indicate head, tail on the 1st coin and head, tail on the 2nd coin. The one-step TPM is given by

$$\begin{array}{c}
\begin{array}{cccc}
& 0 & 1 & O & I \\
\begin{array}{c} 0 \\ 1 \\ O \\ I \end{array} & \begin{bmatrix} 0 & 0 & 1-\beta & \beta \\ 1-\alpha & \alpha & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ 1-\alpha & \alpha & 0 & 0 \end{bmatrix}
\end{array}
\end{array}$$

Let '1' be called as a signal, in applications it might indicate a signal for out of control state of a process. The two coins may correspond to different levels of inspections. For a particular example, one may refer to an illustration given in Unit 7.