

Unit 7

Average Time to Signal in a Sequence of Independent Bernoulli Trials

Abstract

An experiment of tossing of either of the coins A, B according to a specified rule based on success run length is considered. A typical event called 'Signal' is defined based on a sequence of outcomes. An expression for the 'Average Time to Signal' (ATS) is derived by using the renewal theory arguments. Applications of the result in process control are indicated.

1. Introduction

Consider a sequence of outcomes generated in an experiment of tossing of one of the two coins A, B at each unit of time. Let the probabilities of getting tail (failure) on coin A and on coin B be respectively ' α ' and ' β '. For the sequence of outcomes, we define, i^{th} success run length of A as the number of heads on A observed between the $(i-1)^{\text{th}}$ tail (if one such exists) and the i^{th} tail on the coin A including the one at the i^{th} . As a matter of convention, consider the toss at time zero corresponds to a tail on coin A or on coin B. We consider the following experiment.

Description of the experiment:

- (1) Start with tossing of coin A and continue it until tail occurs.
- (2) If X , the corresponding success run length is less than a given integer $L_1 (\geq 2)$ identify the tail as a signal and move back to Step (1). If X is not less than L_1 move to Step (3).
- (3) Start the tossing of coin B and continue it until tail occurs.

(4) If Y , the corresponding success run length on coin B is less than a given integer L_2 (≥ 2) go back to Step (1), else to Step (3).

Let T_i ($i = 1, 2, \dots$) be the inter-occurrence time (in terms of number of tosses) between $(i-1)^{\text{th}}$ and i^{th} signal including i^{th} signal. Note that T_i 's are independently and identically distributed (i.i.d.) random variables (r.v.s). Our problem of interest is to find $E(T_i)$.

2. Notations and an Illustration

Let X_i ($i = 1, 2, \dots$) and Y_j ($j = 1, 2, \dots$) be the i^{th} and j^{th} success run lengths on coin A and B respectively. Note that X_i 's ($i = 1, 2, \dots$) and Y_j 's ($j = 1, 2, \dots$) are independently distributed r.v.s and X_i 's are i.i.d. waiting time r.v.s distributed as X (say) with mean $(1/\alpha)$ and Y_j 's are also i.i.d. waiting time r.v.s distributed as Y (say) with mean $(1/\beta)$ and independent of X_i 's. Let N_A ($= 1, 2, \dots$) and M_A denote the number of tails and the total number of tosses on coin A by the time the first signal is observed. Note that $M_A = \sum_{i=1}^{N_A} X_i$. Observe that X_i ($i = 1, 2, \dots, (M_A-1)$) are not smaller than L_1 and X_{M_A} is less than L_1 . Related to the coin B similar notations are defined, but N_B takes values from zero onwards. It is to be noted that N_A and N_B are not independent. Joint distribution can be obtained by using the arguments similar to those used in the experiment of tossing of two coins described in Unit 6.

The event $\{ N_A = r \text{ and } N_B = s \}$ occurs if and only if $\{ X_1 \geq L_1, X_2 \geq L_1, \dots, X_{(r-1)} \geq L_1, X_r < L_1 \text{ and amongst } Y_1, Y_2, \dots, Y_s \text{ the r.v. } Y_s \text{ is the } (r-1)^{\text{th}} \text{ variable smaller than } L_2 \}$. Since all the r.v.s are independent,

$$P\{ N_A = r \text{ and } N_B = s \} = P\{ X_1 \geq L_1, X_2 \geq L_1, \dots, X_{(r-1)} \geq L_1, X_r < L_1 \} * P\{ \text{amongst } Y_1, Y_2, \dots, Y_s \text{ the r.v. } Y_s \text{ is the } (r-1)^{\text{th}} \text{ variable smaller than } L_2 \}.$$

Hence, for $r = 2, 3, \dots, (s + 1)$ and $s = 2, 3, \dots$, $P(N_A = r, N_B = s)$
 $= P\left\{\bigcap_{i=1}^{(r-1)} (X_i \geq L_1) \bigcap (X_r < L_1)\right\} \cdot P\{\text{amongst } Y_1, Y_2, \dots, Y_s \text{ the r.v. } Y_s \text{ is the } (r-1)^{\text{th}}$
 variable smaller than } L_2 \}

$$= \{b^{(r-1)}(1-b)\} \binom{s-1}{r-2} (1-B)^{(r-1)} B^{(s-r+1)},$$

where $b = P\{X \geq L_1\} = (1-\alpha)^{(L_1-1)}$ and $B = P\{Y \geq L_2\} = (1-\beta)^{(L_2-1)}$.

The joint probability mass functions (p.m.f.) of N_A and N_B is given by,

$$P(N_A = r, N_B = s) = \begin{cases} (1-b) & \text{if } r = 1, s = 0 \\ b(1-b)(1-B) & \text{if } r = 2, s = 1 \\ \binom{s-1}{r-2} [b(1-B)]^{(r-1)} (1-b) B^{s-r+1}, & \text{if } r = 2, 3, \dots, (s+1); \\ & s = 2, 3, \dots \end{cases}$$

Further, it is easy to see that the marginal p.m.f.s of N_A and N_B are as follows.

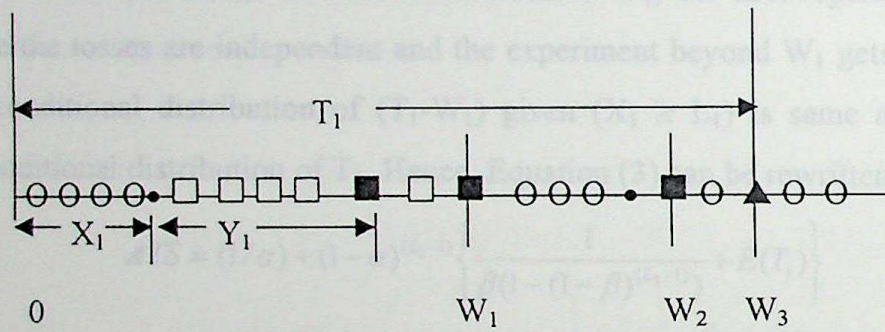
$$P(N_A = r) = b^{(r-1)}(1-b) \quad \text{if } r = 1, 2, \dots$$

$$P(N_B = s) = \begin{cases} (1-b) & \text{if } s = 0 \\ b(1-b)(1-B)\{B + b(1-B)\}^{(s-1)} & \text{if } s = 1, 2, \dots \end{cases}$$

By the definition of T_1 , we have

$$\begin{aligned} T_1 &= \sum_{i=1}^{N_A} X_i + \sum_{j=1}^{N_B} Y_j \\ &= M_A + M_B. \end{aligned}$$

Let $0 = W_0 < W_1 < W_2 < \dots$ be the ordered epochs, so that from $(W_i + 1)^{\text{th}}$ ($i = 0, 1, \dots$) epochs tossing with A restarts. Epochs W_i 's correspond to a tail on B or to a signal. Let $U (= 0, 1, \dots)$ be the number of success runs on coin B before switching from coin B to coin A for the first time. It is to be noted that $0, W_1, W_2, \dots$ are regenerative epochs. For further details one may refer to Smith (1983) or Wolff (1988). As an illustration let $L_1 = 3, L_2 = 4$ and a realization be $X_1 = 5, Y_1 = 5, Y_2 = 2, X_2 = 4, Y_3 = 1, X_3 = 2$. For this realization, the sequence $\{W_k\}$ is given by $\{0, 12, 17, 19, \dots\}$, $U = 2$ and $T_1 = 19$. A diagrammatic representation of the same can be given as follows.



(O: Head on A, $\square$: Head on B, $\bullet$: Tail on A
 $\blacksquare$: Tail on B: $\blacktriangle$: Tail on A with a Signal)

Fig-1: Diagrammatic Representation of the Regenerative Epochs

3. Expression for ATS

To obtain expression for ATS ($= E(T_1)$), we note that

$$\begin{aligned}
 T_1 &= T_1[X_1 < L_1] + T_1[X_1 \geq L_1] \\
 &= X_1[X_1 < L_1] + \{W_1 + T_1 - W_1\}[X_1 \geq L_1] \\
 &= X_1[X_1 < L_1] + W_1[X_1 \geq L_1] + \{T_1 - W_1\}[X_1 \geq W_1] \\
 &= X_1[X_1 < L_1] + \{X_1 + \sum_{j=1}^U Y_j\}[X_1 \geq L_1] + \{T_1 - W_1\}[X_1 \geq L_1]
 \end{aligned}$$

$$= X_1 + \left(\sum_{j=1}^U Y_j \right) [X_1 \geq L_1] + \{T_1 - W_1\} [X_1 \geq L_1] \quad (1)$$

Since X_i 's and Y_j 's are independently distributed r.v.s, taking expectation from Equation (1) we have,

$$E(T_1) = E(X_1) + P\{X_1 \geq L_1\} E\left(\sum_{j=1}^U Y_j\right) + E([X_1 \geq L_1]E(T_1 - W_1 | [X_1 \geq L_1])) \quad (2)$$

$$= (1/\alpha) + (1-\alpha)^{(L_1-1)} E(U)E(Y_j) + E(T_1 - W_1 | X_1 \geq L_1)P\{X_1 \geq L_1\}$$

$$= (1/\alpha) + (1-\alpha)^{(L_1-1)} \left\{ \frac{1}{\beta(1-(1-\beta)^{(L_2-1)})} \right\} + E(T_1 - W_1 | X_1 \geq L_1)(1-\alpha)^{(L_1-1)} \quad (3)$$

When $(X_1 \geq L_1)$, it is certain to observe W_1 , the first regenerative epoch. Since the tosses are independent and the experiment beyond W_1 gets regenerated, the conditional distribution of $(T_1 - W_1)$ given $(X_1 \geq L_1)$ is same as that of the unconditional distribution of T_1 . Hence, Equation (3) can be rewritten as

$$ATS = (1/\alpha) + (1-\alpha)^{(L_1-1)} \left\{ \frac{1}{\beta(1-(1-\beta)^{(L_2-1)})} + E(T_1) \right\}$$

and hence,

$$ATS = \{1 - (1-\alpha)^{(L_1-1)}\}^{-1} \left\{ \frac{(1-\alpha)^{(L_1-1)}}{\beta(1-(1-\beta)^{(L_2-1)})} + \frac{1}{\alpha} \right\}. \quad (4)$$

In the following section, we discuss an application of the result in process control.

4. An Application to a Two Level Process Control

The above sequence of tossing of coins arise naturally in a production process, with tossing of coins corresponding to the inspections and the results 'Head' and 'Tail' corresponding to conforming and non-conforming to the quality of the product based on the inspections. Further, a 'Signal' indicates 'Out of control' status of the process.

As a particular example, tossing of first coin refer to the unit level inspection and that of the coin B correspond to inspection of a group of n units. Suppose p (q) and P (Q) are the probabilities of a unit being defective (non-defective) and a group being non-conforming (conforming) to the quality.

Note that M_A and M_B are respectively the number of units and the groups inspected by the time the process is declared as out of control. Hence the total number of units inspected by the time the process declared as out of control is $M_A + nM_B$. Thus an expression for the average number of units inspected by the time the process is declared as out of control is given by

$$E(T_1) = E(M_A + nM_B).$$

Hence from Equation (2), we have,

$$\begin{aligned} E(T_1) &= E(X_1) + P\{X_1 \geq L_1\}nE(M_B) + E([X_1 \geq L_1]E(T_1 - W_1 | [X_1 \geq L_1])) \\ &= (1/p) + nq^{(L_1-1)}E(U)E(Y_j) + E(T_1 - W_1 | X_1 \geq L_1)P\{X_1 \geq L_1\} \\ &= (1/p) + nq^{(L_1-1)}\left\{\frac{1}{P(1-Q^{(L_2-1)})}\right\} + E(T_1 - W_1 | X_1 \geq L_1)q^{(L_1-1)} \\ &= (1/p) + q^{(L_1-1)}\left\{\frac{n}{P(1-Q^{(L_2-1)})} + E(T_1)\right\} \end{aligned}$$

and hence,

$$E(T_1) = \{1 - q^{(L_1-1)}\}^{-1} \left\{ \frac{nq^{(L_1-1)}}{P(1-Q^{(L_2-1)})} + \frac{1}{p} \right\}.$$