

Unit 2

Models Based on Composite Functions**Abstract**

If F is any distribution function (d.f.) and H is a d.f. on $[0, 1]$, then $G(x) = H(F(x))$ is a d.f.. Let $\mathcal{F}$ be a family of distribution functions indexed by θ and $\mathcal{H}$ be a family of distribution functions defined on $[0, 1]$ and indexed by α . Then by using the composite function $H.F$, an extended family of distributions can be defined. By using suitable family H desired structural changes in the family $\mathcal{F}$ can be achieved. Some class properties of the families of distributions are defined.

1. Notations and Preliminaries

Let $\mathcal{F} = \{F(x, \theta); \theta \in \Theta\}$ be a given family of d.f.s and $\mathcal{H} = \{H(x, \alpha); \alpha > 0\}$ be a class of d.f.s defined on $[0, 1]$ with $H(0+, \alpha) = 0$, $H(1-, \alpha) = 1$ and $H(x, 1) = x$.

Then $G(x, \theta, \alpha) = H(F(x, \theta), \alpha)$ is a proper d.f. By using $\mathcal{H}$, a given family $\mathcal{F}$ of d.f.s can be extended to $\mathcal{G} = \{G(x, \theta, \alpha), \theta \in \Theta, \alpha > 0\}$. If H and G are symmetric then G is symmetric about the point of symmetry of F .

Following are some relevant questions :

- (i) Is $\mathcal{G}$ a nontrivial extension?
- (ii) Can (θ, α) be considered as parameter for the family $\mathcal{G}$?

We note that in general $\mathcal{F} \subset \mathcal{G}$, however $\mathcal{F}$ and $\mathcal{G}$ may be identical. For example $H(x, \alpha) = x^\alpha$ and $F(x, \theta) = x^\theta$, for $0 \leq x \leq 1$ and $\theta, \alpha \geq 0$. Then,

$$G(x, \alpha, \theta) = (x^\theta)^\alpha = x^{\theta\alpha},$$

thus $\mathcal{G} = \mathcal{F}$ and (α, θ) is not parameter for the family $\mathcal{G}$.

In sub section 2.2 we define some closure properties of the $\mathcal{H}$.

1.2. Some Class Properties of $\mathcal{H}$:

a) Class $\mathcal{H} = \{H(x, \alpha); \alpha > 0\}$ is said to be *closed under composition* if for $\alpha_1, \alpha_2 \in \mathbb{R}^+ \exists \alpha^* \in \mathbb{R}^+$, such that $H(H(x, \alpha_2), \alpha_1) = H(x, \alpha^*) \forall 0 \leq x \leq 1$.

b) We say that a class $\mathcal{H}$ is *closed under inverse operation* if for each $\alpha \in \mathbb{R}^+ \exists \alpha^* \in \mathbb{R}^+$, such that

$$H(H(x, \alpha^*), \alpha) = H(H(x, \alpha), \alpha^*) = x, 0 \leq x \leq 1.$$

c) Let $H \in \mathcal{H}$ then

(i) $H_D(x, \alpha) = 1 - H(1 - x, \alpha)$ is called the Dual of H

(ii) H is said to be self dual if $H_D(x, \alpha) = H(x, \alpha)$.

If H is self dual then $H(1/2, \alpha) = 1/2$.

d) A class $\mathcal{H}$ is said to be *closed under duality* if for each $H \in \mathcal{H}, H_D \in \mathcal{H}$ i.e. for each $\alpha \in \mathbb{R}^+$ there exists $\alpha^* \in \mathbb{R}^+$ such that

$$H(x, \alpha^*) = H_D(x, \alpha).$$

Further if $\alpha^* = \alpha, \forall \alpha \in \mathbb{R}^+$, then the class $\mathcal{H}$ is said to be a *self dual class*.

Examples:

i) $H(x, \alpha) = x^\alpha,$

$$H_D(x, \alpha) = 1 - (1 - x)^\alpha$$

With $\alpha = n$, the functions H and H_D respectively correspond to d.f. of maximum and minimum of a random sample of size n from $U(0, 1)$.

ii) $H(x, \alpha) = x / \{\alpha + (1 - \alpha)x\}$

$$H_D(x, \alpha) = \alpha x / \{1 - (1 - \alpha)x\}$$

iii)
$$H(x, \alpha) = \begin{cases} \frac{\alpha^x - 1}{\alpha - 1}, & \alpha \neq 1 \\ x, & \alpha = 1 \end{cases}$$

$$H_D(x, \alpha) = H(x, 1/\alpha)$$

$$\text{iv)} \quad H(x, \alpha) = \frac{H^*(2x)}{2} \left[x < \frac{1}{2} \right] + \left\{ 1 - \frac{H^*(2(1-x))}{2} \right\} \left[x > \frac{1}{2} \right]$$

where $H^*(x, \alpha)$ is any member of $\mathcal{H}$.

$$H_D(x, \alpha) = H(x, \alpha).$$

Following table shows whether the classes of functions defined in examples i), ii), iii) and iv) are closed under composition/ inverse/ duality/ self duality.

Example	Composition	Inverse	Dual	Self dual
1	Yes	Yes	No	No
2	Yes	Yes	Yes	No
3	No	Yes	Yes	No
4	Yes if F^* is closed under composite	Yes if F^* is Closed under inverse	Yes	Yes

We note that all the functions $H(x, \alpha)$, $0 \leq x \leq 1$ defined in i) to iv) are proper d.f.s. for each $\alpha \in \mathbb{R}^+$. Thus if $F(x)$ is a proper d.f., then for each $\alpha \in \mathbb{R}^+$,

$$G(x, \alpha) = H(F(x), \alpha) \text{ and } K(x, \alpha) = 1 - H(1 - F(x), \alpha) \text{ are proper d.f.s.}$$

It is to be noted that if F is a self dual class (that is $H(x, \alpha) = 1 - H(1 - x, \alpha)$), then for each $\alpha \in \mathbb{R}^+$,

$$K(x, \alpha) = 1 - H(1 - F(x), \alpha) = H(F(x), \alpha) = G(x, \alpha).$$

Mudholkar et al.(1995) Have the h function defined in (1) to generate the Weibull family. Marshall and Olkin (1997) have used h function of example (ii) to extend the class of exponential distributions.

2. Models based on reliability functions:

If H and F are differentiable and admit densities h and f respectively then both G and K admit densities respectively given by

$$g(x, \alpha) = h(F(x), \alpha) f(x)$$

$$\text{and } k(x, \alpha) = h(1 - F(x), \alpha) f(x)$$

The failure rate function r_K corresponding to K is given by

$$\begin{aligned} r_K(x) &= \frac{h'(1 - F(x), \alpha) f(x)}{K(x, \alpha)} \\ &= \frac{h'(1 - F(x), \alpha) (1 - F(x))}{H(1 - F(x), \alpha)} r_f(x) \quad \dots\dots\dots(1) \end{aligned}$$

(i) If $r_K(x)$ is proportional $r_f(x)$ then from (1), $\frac{h(y, \alpha)}{H(y, \alpha)} y = m(\alpha)$ (say) is not a function of x . Thus,

$$-\log H(x, \alpha) = m(\alpha) \log x + c.$$

But as $H(1, \alpha) = 1$, $H(0, \alpha) = 0$ we have $c = 0$. Hence we have $H(x, \alpha) = x^{k(\alpha)}$ which is Lehmann family of distributions. Equivalently corresponds to Cox's Proportional Hazards Model.

ii) If $\frac{f'(x, \alpha)}{f(x, \alpha)} x = \frac{x \log \alpha}{\alpha^{x-1} - 1}$ then $\log f(x, \alpha) = \log(\alpha^{-1} - \alpha^x)$.

But as $h(1, \alpha) = 1$ we have $c = \log(\alpha/1-\alpha)$

Hence, $h(x, \alpha) = \alpha^{1-x} - 1/(\alpha - 1)$.