

Unit 3

A Class of Drifted Distributions

Abstract

Let F be a given distribution function (d.f.) with $0 < F(0) < 1$. For $a \geq F(0-)$ define a d.f.

$$G(x, a) = \begin{cases} F(ax)/a & \text{for } x < 0 \\ 1 - \frac{1 - F(bx)}{b} & \text{for } x \geq 0, \end{cases}$$

where $b = \frac{1 - F(0)}{1 - F(0-)/a}$. We note that when $F(0) = F(0-)$, $G(x, 1) = F(x)$.

Let G be called a *drifted* distribution function obtained from F and a be called the *drift parameter*. Distributional properties of G , when F is symmetric about zero have been studied. Some related inference problems for the family $\{G(x, a) : a > \frac{1}{2}\}$ have been considered when F corresponds to Standard Laplace and Standard Normal distribution, by introducing the location and scale parameters.

1. Introduction :

In the literature, classes of distributions have been introduced by considering typical functions of distribution functions. For example, Mudholkar, Srivastava and Freimer (1995) have used exponentiated function, Marshall & Olkin (1997) have used a ratio of two linear functions to extend the class of exponential family of distributions.

In this article, to extend a family of distributions, we propose a different kind of technique by using a '*drift*' parameter and later the location and scale parameters are introduced. In particular, we consider extensions of symmetric distributions and study distributional properties of such a family. The proposed

approach is different from the one considered by Jones and Faddy (2003). "The epsilon-skew-normal distribution for analyzing near-normal data" by Mudholkar et al (2000) is a particular example.

We consider the problems of estimation and testing for the extended three-parameter family obtained by Laplace and Normal distributions. These problems are different from those considered by John (1982) and Nabar and Barpande (2001). Extensive simulation study and data analysis using the proposed models has been carried out.

2. Properties of Drifted Distribution :

2.1 Nature of drifted distribution function:

Consider the class of exponential distributions with location parameter $\mu = -1$ and $\sigma = 1$. That is,

$$F(x, -1, 1) = \begin{cases} 0 & \text{if } x \leq -1, \\ 1 - \exp\{-(x+1)\} & \text{if } -1 < x. \end{cases}$$

Define,

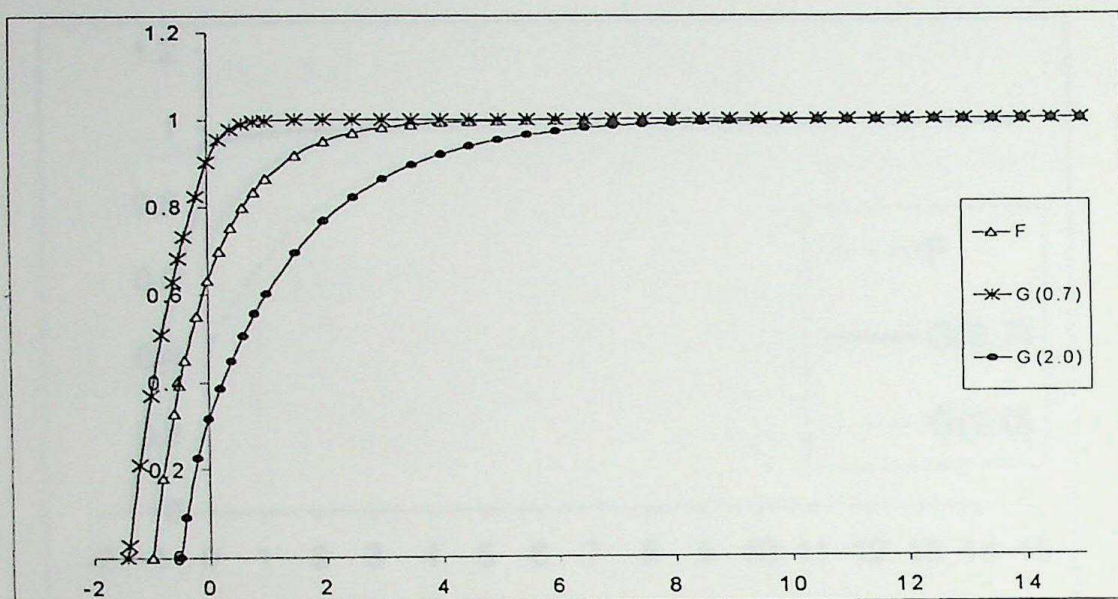
$$G(x, a) = \begin{cases} 0 & \text{if } ax \leq -1, \\ \frac{1}{a}(1 - \exp\{-(ax+1)\}) & \text{if } -1 < ax < 0, \\ \frac{1}{b}(1 - \exp\{-(bx+1)\}) & \text{if } ax \geq 0. \end{cases}$$

where $b = \frac{1-F(0)}{1-F(0-)/a}$. We note that G is a d.f. for $b \geq \frac{1-F(0)}{1-F(0-)/a}$ and

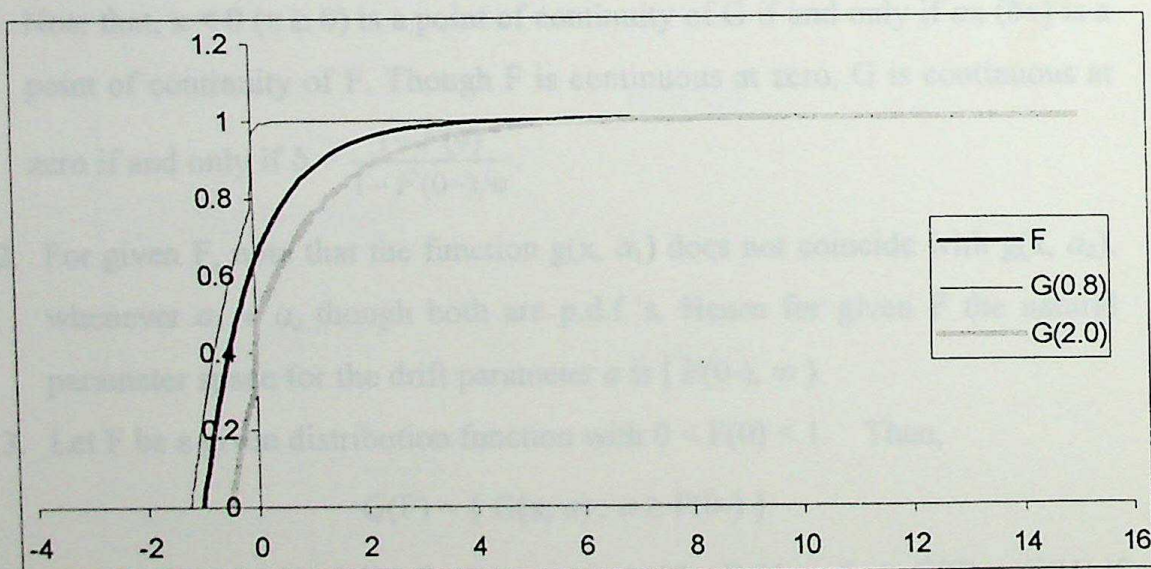
if $b > \frac{1-F(0)}{1-F(0-)/a}$, then $G(0) - G(0-) > 0$, that is G has a jump point at $\mu = -1$.

The graphs of functions F when $\mu = -1$, $\sigma = 1$ and G with $a = 0.7, 2$ and

$b = \frac{1-F(0)}{1-F(0-)/a}$ are given below:-



Also, the graphs of functions F and G with $a = 0.8, 2$ and $b > \frac{1-F(0)}{1-F(0-)/a}$ are given below. In this case, the drifted functions have jumps at 0.

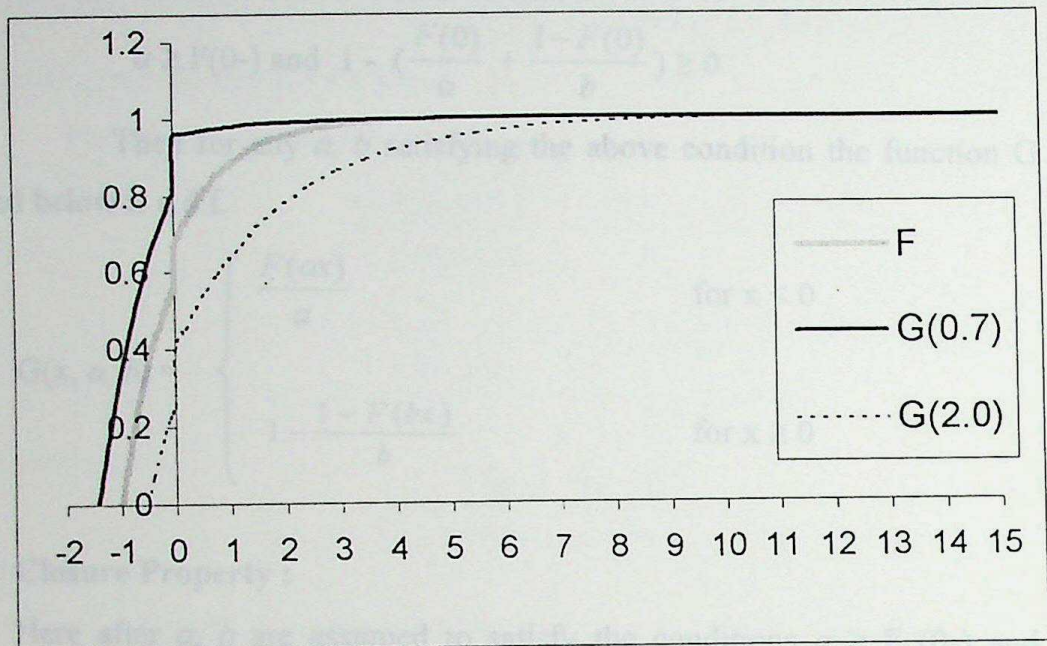


If F has a jump point then G also has a jump point. Consider the d.f.

$$F(x) = (1-\delta)[1-e^{-(x+1)}] + \delta[x \geq 0], \quad x \geq -1. \text{ We note that } F \text{ has a jump at}$$

zero. The graphs of the functions F, G with $a = 0.7, 2$ and $b = \frac{1-F(0)}{1-F(0-)/a}$

are as follows :



Remarks:

1. Note that, $x < 0$ ($x \geq 0$) is a point of continuity of G if and only if ax (bx) is a point of continuity of F . Though F is continuous at zero, G is continuous at zero if and only if $b = \frac{1 - F(0)}{1 - F(0-)/a}$.

2. For given F , note that the function $g(x, a_1)$ does not coincide with $g(x, a_2)$, whenever $a_1 \neq a_2$ though both are p.d.f.'s. Hence for given F the natural parameter space for the drift parameter a is $[F(0-), \infty)$.

3. Let F be a given distribution function with $0 < F(0) < 1$. Then,

$$G(F) = \{ G(x, a) : a \geq F(0-) \}$$

is a family of drifted distributions generated by F . Note that, $G(F) = G(H)$ if and only if $H = G(x, a)$ for some a , equivalently, if and only if $H \in G(F)$.

4. For $x < 0$, $G(x, a)$ is decreasing function of a and decreases to zero as a increases to ∞ .
5. A model with two drift parameters can be given as follows :

Let F be any d.f. and a, b be such that

$$a \geq F(0-) \text{ and } 1 - \left(\frac{F(0)}{a} + \frac{1-F(0)}{b} \right) \geq 0$$

Then for any a, b satisfying the above condition the function G defined below is a d.f.

$$G(x, a, b) = \begin{cases} \frac{F(ax)}{a} & \text{for } x < 0 \\ 1 - \frac{1 - F(bx)}{b} & \text{for } x \geq 0 \end{cases}$$

2.2 Closure Property :

Here after a, b are assumed to satisfy the conditions $a \geq F(0-)$ and $\frac{F(0-)}{a} + \frac{1-F(0)}{b} = 1$. To be specific we shall write $G(x, a)$ as $G(F, a)(x)$ to mean the drifted distribution by using F and the drift parameter a . Consider

$$\begin{aligned} & G(G(F, a_1), a_2)(x) \\ &= \begin{cases} \frac{G(F, a_1)(a_2 x)}{a_2} & \text{for } x < 0 \\ 1 - \frac{1 - G(F, a_1)(b_2 x)}{b_2} & \text{for } x > 0 \end{cases} \\ &= \begin{cases} \frac{F(a_1 a_2 x)}{a_1 a_2} & \text{for } x < 0 \\ 1 - \frac{1 - \left[1 - \frac{1 - F(b_1 b_2 x)}{b_1} \right]}{b_2} & \text{for } x > 0 \end{cases} \\ &= \begin{cases} \frac{F(a_1 a_2 x)}{a_1 a_2} & \text{for } x < 0 \\ 1 - \frac{1 - F(b_1 b_2 x)}{b_1 b_2} & \text{for } x > 0 \end{cases} \end{aligned}$$

which is equal to $G(F, a_1 a_2)(x)$ provided $\frac{F(0-)}{a_1 a_2} + \frac{1-F(0)}{b_1 b_2} = 1$ and in the following we shall prove the same.

The relation between b_1 and b_2 is

$$\begin{aligned} \frac{G(F, a_1)(0-)}{a_2} + \frac{1 - G(F, a_1)(0)}{b_2} &= 1 \\ \text{i.e. } \frac{F(0-)/a_1}{a_2} + \frac{1 - \left[1 - \frac{1-F(0)}{b_1} \right]}{b_2} &= 1 \\ \text{i.e. } \frac{F(0-)}{a_1 a_2} + \frac{1-F(0)}{b_1 b_2} &= 1. \end{aligned}$$

Hence,

$$G(G(F, a_1), a_2)(x) = G(F, a_1 a_2)(x) \quad (1)$$

It is to be noted that in the above defined composition, $a_1 \geq F(0-)$ and $a_2 \geq G(F, a_1)(0-) = F(0-)/a_1$. And hence, $a_2 = 1/a_1 (> F(0-)/a_1)$ is a permissible value..

By taking $a_2 = \frac{1}{a_1}$, we have,

$$G(G(F, a_1), 1/a_1)(x) = G(F, 1)(x) = F.$$

Thus the family $\{G(F, a)(x) : a \geq F(0-)\}$ is closed under the composition defined in (1) with $G(F, 1)$ as unity.

Further, the inverse of $G(F, a)(x)$ is $G(F, a^{-1})(x)$.

That is the family $\{G(F, a) : a \geq F(0-)\}$ is a group of drifted distributions generated by F , closed under the composition defined in (1) with F as the identity element.

2.3 Class generated by symmetric distributions :

Consider the family

$$G = \{G(F, a) : F \text{ Symmetric about } 0 \text{ and } a \geq 1/2 (= F(0))\} \quad \text{---- (2)}$$

Let F_1 and F_2 be symmetric about 0 and

$$G(F_1, a_1) \quad G(F_2, a_2) \quad \text{---- (3)}$$

In particular, at 0 we have $\frac{F_1(0)}{a_1} = \frac{F_2(0)}{a_2}$, implies $\frac{1}{2a_1} = \frac{1}{2a_2}$.

Hence, $a_1 = a_2 = a$ (say).

Further from (3), $\frac{F_1(ax)}{a} = \frac{F_2(ax)}{a}; \forall x < 0$

Thus $F_1(x) = F_2(x)$ for $x < 0$. Similarly $F_1(x) = F_2(x)$ for $x \geq 0$.

Hence, $\{(F, a) : F \text{ symmetric about } 0, a \geq 1/2\}$ constitute the parameter space for the family $\mathcal{G}$ defined (2).

Illustrations :

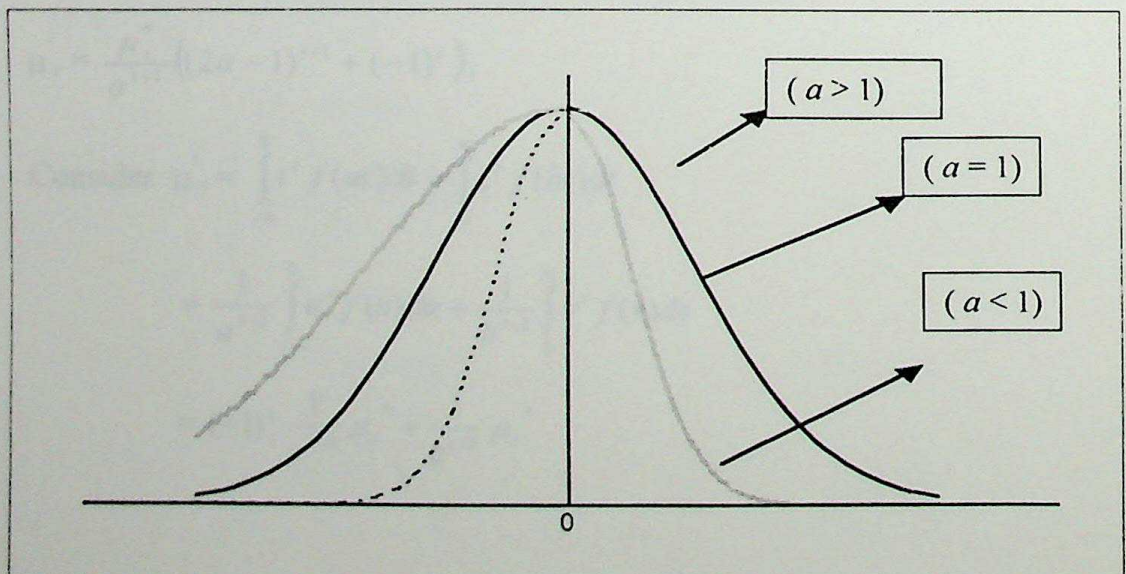
Let F be differentiable find $\frac{dF}{dx} = f$ be symmetric about 0. Let $\mathcal{F}$ be the class of all symmetric density functions symmetric about 0. Then for $a > 1/2$ ($= F(0)$), define a density function

$$g(x, a) = \begin{cases} f(ax) & \text{for } x < 0, \\ f(bx) & \text{for } x > 0, \end{cases}$$

where $b = (2 - 1/a)^{-1} = \frac{a}{2a-1}$.

Note that $g(x, 1) = f(x)$ and $\mathcal{G} = \{g(x, a) : a > 1/2, f \in \mathcal{F}\}$ is an extended class of drifted distributions derived from $\mathcal{F}$

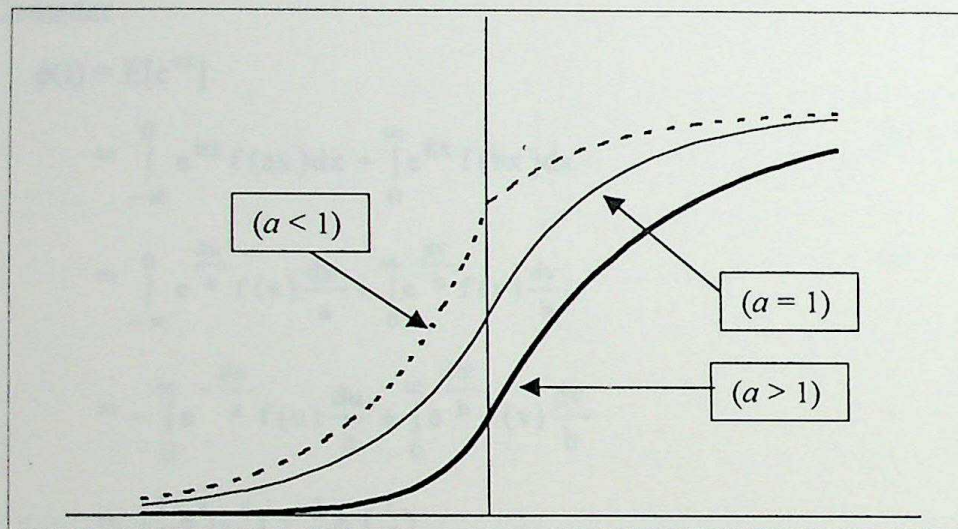
The graphs of the function g for $a > 1$, $a = 1$ and $a < 1$ are as follows:



Then G , the corresponding distribution function is given by

$$G(x, a) = \begin{cases} \frac{F(ax)}{a} & \text{for } x \leq 0 \\ \frac{1}{2a} + \left[F(bx) - \frac{1}{2} \right] \left(2 - \frac{1}{a} \right) & \text{for } x > 0. \end{cases}$$

The graphs of function G for $a > 1$, $a = 1$ and $a < 1$ are as follows:



2.4 Properties of the distribution G :

(i) Moments : the r^{th} moment of distribution G is given by

$$\mu_r' = \frac{\mu_r^*}{a^{r+1}} \left((2a-1)^{r+1} + (-1)^r \right),$$

$$\begin{aligned} \text{Consider } \mu_r' &= \int_{-\infty}^0 t^r f(at) dt + \int_0^{\infty} t^r f(bt) dt \\ &= \frac{1}{a^{r+1}} \int_{-\infty}^0 u^r f(u) du + \frac{1}{b^{r+1}} \int_0^{\infty} v^r f(v) dv \\ &= (-1)^r \frac{1}{a^{r+1}} \mu_r^* + \frac{1}{b^{r+1}} \mu_r^* \end{aligned}$$

$$= \frac{\mu_r^*}{a^{r+1}} ((2a-1)^{r+1} + (-1)^r)$$

In particular, $\mu_1^* = \frac{4(a-1)}{a} \mu_1^*$ and $M_a = \frac{2a-1}{a} F^{-1}\left(\frac{3a-2}{2(2a-1)}\right)$.

(ii) Characteristic function : $\phi(t) = -\frac{1}{a} \phi^*\left(-\frac{t}{a}\right) + \frac{1}{b} \phi^*\left(\frac{t}{b}\right)$

where $\phi^*(t) = \int_0^\infty e^{itx} f(x) dx$.

Consider

$$\begin{aligned} \phi(t) &= E[e^{itx}] \\ &= \int_{-\infty}^0 e^{itx} f(ax) dx + \int_0^\infty e^{itx} f(bx) dx \\ &= \int_{-\infty}^0 e^{\frac{itu}{a}} f(u) \frac{du}{a} + \int_0^\infty e^{\frac{itv}{b}} f(v) \frac{dv}{b} \\ &= -\int_0^\infty e^{-\frac{itu}{a}} f(u) \frac{du}{a} + \int_0^\infty e^{\frac{itv}{b}} f(v) \frac{dv}{b} \\ &= -\frac{1}{a} \phi^*\left(-\frac{t}{a}\right) + \frac{1}{b} \phi^*\left(\frac{t}{b}\right). \end{aligned}$$

(iii) If $X \sim G_a$, then $-X \sim G_b$

Consider

$$\begin{aligned} P[-X < t] &= 1 - G_a(-t) \\ &= \begin{cases} 1 - \frac{1}{a} F(-at) & \text{for } t > 0 \\ 1 - \left[\frac{1}{2a} + \left\{ F(-bt) - \frac{1}{2} \right\} \frac{1}{b} \right] & \text{for } t < 0 \end{cases} \\ &= \begin{cases} 1 - \frac{1}{a} (1 - F(at)) & \text{for } t > 0 \\ \frac{2a-1}{2a} - \frac{F(-bt)}{b} + \frac{1}{2b} & \text{for } t < 0 \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{a-1}{a} + \frac{1}{a} F(at) & \text{for } t > 0 \\ \frac{1}{b} [1 - F(-bt)] & \text{for } t < 0 \end{cases}$$

$$= \begin{cases} \frac{2a-1}{2a} - \frac{1}{2a} + \frac{F(at)}{a} & \text{for } t > 0 \\ \frac{1}{b} F(bt) & \text{for } t < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2b} + \left\{ F(at) - \frac{1}{2} \right\} \frac{1}{a} & \text{for } t > 0 \\ \frac{1}{b} F(bt) & \text{for } t < 0 \end{cases}$$

$$= G_b(t).$$

(iv) **Stochastic ordering** : Let f be unimodal with 0 as the modal value and

$$a_1 < a_2. \text{ Then } X_{a_1} \stackrel{St}{\leq} X_{a_2}.$$

Since f is unimodal, f increases if $x < 0$ and f is decreasing if $x > 0$.

Case (I) : $x < 0$. Since f is increasing for $x < 0$ and $0 < a_1 < a_2$, we have

$$f(a_1 x) \geq f(a_2 x). \text{ and hence, } G(t, a_1) = \int_{-\infty}^t f(a_1 x) dx$$

$$\geq \int_{-\infty}^t f(a_2 x) dx$$

$$= G(t, a_2), \quad \text{for } x < 0.$$

Case (II) : $x \geq 0$ For $a_1 < a_2$, we have $b_1 > b_2$ $f(b_1 x) < f(b_2 x)$ and for

$$t > 0, \quad \overline{G}(t, a_1) = \int_t^{\infty} f(b_1 x) dx < \int_t^{\infty} f(b_2 x) dx = \overline{G}(t, a_2). \text{ That is for } x \geq 0.$$

$G(t, a_1) \geq G(t, a_2)$. Thus we have $G(t, a_1) \geq G(t, a_2), \forall t$. Hence, for $a_1 < a_2$,

$$X_{a_1} \stackrel{st}{\leq} X_{a_2}.$$

In particular, for $\frac{1}{2} < a < 1$, $G(t, a) > G(t, 1) = F(t), \forall t \in \mathfrak{R}$ and for $a \geq 1$, $G(t) > F(t) \quad \forall t \in \mathfrak{R}$.

3. Problem of estimation :

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from $G \in \mathcal{G}$. In the following we consider the problem of estimation of the drift parameter when A) F is symmetric about 0 and B) F corresponds to standard normal distribution.

3.1 Estimator of a when F is symmetric about 0 :

(i) *Estimation of 'a' when F is unknown symmetric distribution symmetric about 0 (based on a quantile) :*

We know that $G(0, a) = 1/2a$, for all $G \in \mathcal{G}$. Hence, a simple estimator of a can be obtained by equating $1/2a$ to the proportion P_n of the negative observations. That is, $\tilde{a} = 1/2P_n$. Note that $nP_n \sim B(n, 1/2a)$, the estimator $\tilde{a}$ is a non-negative extended real-valued random variable with $P(a = \infty) = (1/2b)^n \rightarrow 0$ as $n \rightarrow \infty$ (since, $b < 1$).

(ii) *Estimation of 'a' when F is a known symmetric distribution symmetric about 0 (based on moments) :*

By equating the population mean, $\mu_1 (= \frac{4(a-1)}{a} \mu_1^*)$

to $\bar{x}$, we get, $\bar{a} = \frac{1}{1 - \frac{\bar{x}}{4\mu_1^*}}$, where $\mu_1^* = \int_0^\infty t f(t) dt$.

For example, for drifted distribution derived from Standard Normal, $\bar{a} = \frac{4}{4 - \sqrt{2\pi} \bar{x}}$ and for drifted distribution derived from

Laplace distribution, $\bar{a} = \frac{1}{1 - \frac{\bar{x}}{2}}$.

3.2 A family of drifted distributions admitting two-dimensional sufficient statistic:

Let $Y_1, \dots, Y_n$ be iid with common d.f. $G(x, a)$ and F be a specified symmetric distribution about 0. Further the pdf f corresponding to F be such that for every $x_1, \dots, x_r < 0$,

$$\prod_{i=1}^r g(x_i, a) = \prod_{i=1}^r f(ax_i) = h(a, t(y_1, \dots, y_r)).$$

Let $Y_{(1)}, \dots, Y_{(r)}, Y_{(r+1)}, \dots, -Y_{(n)}$ be order statistics and $Y_{(r)} < 0 < Y_{(r+1)}$ for some $r = 0, 1, \dots, n$. In the following, we show that when f has the above form $U = t(Y_{(1)}, \dots, Y_{(r)})$ and $V = t(-Y_{(r+1)}, \dots, -Y_{(n)})$ are sufficient for a .

Let $f_1(y) = f(y | y < 0)$ and $f_2(y) = f(y | y > 0)$, then $f_1(y) = f_2(-y)$ and further the functions f_1 and f_2 are such that $f_1(ay_1, \dots, ay_r) = h(a, t(y_1, \dots, y_r))$ for $y_1, \dots, y_r < 0$. We shall have the convention that if $r = 0$, then $f_1(\cdot) = 1$ and if $r = n$, then $f_2(\cdot) = 1$.

$$\begin{aligned} \text{Consider } f_2(by_{r+1}, \dots, by_n) &= \prod_{i=r+1}^n \frac{f(by_i)[y_i > 0]}{P(Y_i > 0)} \\ &= \prod_{i=r+1}^n \frac{f(-by_i)[-by_i < 0]}{1/2} \\ &= \prod_{i=r+1}^n \frac{f(b, z_i)[-z_i < 0]}{1/2} \\ &= h(b, t(-y_{r+1}, \dots, -y_n)) \end{aligned}$$

Hence,

$$\begin{aligned}
 g(y_1, \dots, y_n) &= \frac{n!}{2^n} f_1(y_{(1)}, \dots, y_{(r)}) f_2(y_{(r+1)}, \dots, y_{(n)}) \\
 &= \frac{n!}{2^n} h(a, t(y_{(1)}, \dots, y_{(r)})) h(b, t(-y_{(r+1)}, \dots, -y_{(n)})) \\
 &= \frac{n!}{2^n} h(a, U) h(b, V)
 \end{aligned}$$

Thus, (U, V) is sufficient for a .

For example, pdf's corresponding to standard normal and Laplace distributions have above forms. However, pdf corresponding to Cauchy does not have this form.

For example, in case of drifted distribution derived from Standard Normal distribution, $h(a, t) = \exp(-a^2 t / 2)$. In case of drifted distribution derived from Laplace distribution, $h(a, t) = \exp(-a|t|)$.

4. Drifted Standard Laplace Distribution :

The pdf corresponding to drifted distribution obtained from Standard Laplace distribution is given by

$$g(x, a) = \begin{cases} \frac{1}{2} e^{ax} & \text{for } x < 0 \\ \frac{1}{2} e^{-bx} & \text{for } x > 0, \end{cases}$$

where $a > 1/2$ and $\frac{1}{a} + \frac{1}{b} = 2$.

Remarks :

1. The p.d.f. $g(x, a)$ can be expressed as

$$g(x, a) = \frac{1}{2} \exp \left\{ \left(x \frac{a}{2a-1} \right) [2aI(x) - 1] \right\}, \text{ where } I(x) = 1 \text{ if } x < 0$$

and zero otherwise. Hence, this distribution does not belong to one parameter exponential family. However, it belongs to Cramer family.

2. The drifted distribution obtained from $L(0, \sigma)$ belongs to two parameter exponential family.

4.1 Some Distributional Properties :

$$(i) \mu_r = \frac{r+1}{2a^{r+1}} \left[(2a-1)^{r+1} + (-1)^r \right].$$

(We know that, $\mu_r = \frac{\mu_r^*}{a^{r+1}} \left[(2a-1)^{r+1} + (-1)^r \right]$ and in this case, $\mu_r^* = \frac{r+1}{2}$.)

$$(ii) \text{Mean} = \left(\frac{1}{b} - \frac{1}{a} \right) \text{ and Variance} = 4 - \frac{2}{ab}.$$

$$(iii) \text{Characteristic function is } \phi(t) = \frac{1}{2(a+it)} + \frac{1}{2(b-it)}.$$

- (iv) The Fisher information contained in a single observation :

Differentiating the logarithm of pdf w.r.t. a we get

$$\frac{\partial \log g}{\partial a} = \begin{cases} x & \text{for } x < 0 \\ \frac{-x}{(2a-1)^2} & \text{for } x > 0 \end{cases}$$

$$E \left[\frac{\partial \log g}{\partial a} \right]^2 = \frac{1}{2} \int_{-\infty}^0 x^2 e^{ax} dx + \frac{1}{2} \int_0^{\infty} \frac{(-x)^2}{(2a-1)^4} e^{-bx} dx$$

$$\text{i.e. } I(a) = \frac{2}{a^2(2a-1)}.$$

4.3 Estimation of the drift parameter a :

Consider the problem of estimation of the drift parameter a when the distribution is derived from the standard Laplace distribution.

Let r of the n random observations from g be negative and remaining $n-r$ be positive. Let $U = \sum_{x_i < 0} |x_i|$ and $V = \sum_{x_i \geq 0} |x_i|$. The

loglikelihood function is given by

$$\text{Log } L = -n \log 2 - [aU + bV].$$

Case (a) If $r = 0$, i.e. $U = 0$, then $\text{Log } L = -n \log 2 - [bV]$ which is a decreasing function of b and hence increasing function of a . Hence, mle of a does not exist. However in this case, 'n' can be taken as a modified estimate of a .

Case (b) If $r = n$, i.e. $V = 0$, $\text{Log } L = -n \log 2 - [aU]$ which is a decreasing function of a . Hence, mle of a is $\hat{a} = \frac{1}{2}$.

$$\text{Case (c) For } 0 < r < n, \quad \frac{\partial \log L}{\partial a} = 0$$

$$\Rightarrow \left[U + V \frac{db}{da} \right] = 0.$$

The unique solution of the above equation is $\hat{a} = \frac{1}{2} \left[1 + \left(\frac{V}{U} \right)^{\frac{1}{2}} \right]$.

Hence, for $0 < r < n$, mle of a is $\hat{a} = \frac{1}{2} \left[1 + \left(\frac{V}{U} \right)^{\frac{1}{2}} \right]$.

5. Drifted Laplace Distribution with location and scale parameter :

By introducing location and scale parameters in a drifted distribution obtained from standard Laplace distribution, we get the following pdf

$$g(x, \theta, \sigma, a) = \begin{cases} \frac{1}{2\sigma} \exp\left\{ \frac{a}{\sigma}(x - \theta) \right\} & \text{for } x < \theta \\ \frac{1}{2\sigma} \exp\left\{ -\frac{b}{\sigma}(x - \theta) \right\} & \text{for } x \geq \theta. \end{cases}$$

5.1 Estimation of the parameters :

(I) Estimation of a and θ when σ is known :

Consider the problem of estimation of a and θ when σ is known.

Without loss of generality, let $\sigma = 1$.

Let $x_{(1)} < \dots < x_{(n)}$ be the observed ordered values and for notational convenience define $x_{(0)} = -\infty$ and $x_{(n+1)} = \infty$. The likelihood function is given by

$$L(a, \theta) = \frac{1}{2^n} \exp \left\{ a \sum_{i=1}^r (x_{(i)} - \theta) - b \sum_{i=r+1}^n (x_{(i)} - \theta) \right\}$$

for $x_{(r)} < \theta \leq x_{(r+1)}$ and $r = 0, 1, \dots, n$.

Let be a fixed. We note that for $\theta \leq x_{(1)}$,

$$L(a, \theta) = \frac{1}{2^n} \exp \{-bn(\bar{x} - \theta)\}$$

is an increasing function of θ for each a fixed. Further, for $\theta > x_{(n)}$

$$L(a, \theta) = \frac{1}{2^n} \exp\{an(\bar{x} - \theta)\}$$

is a decreasing function of θ for each a fixed. Hence to find m.l.e. of (a, θ) it is enough if we confine to the domain $\{(a, \theta) : a \geq 1/2, x_{(1)} \leq \theta \leq x_{(n)}\}$. For a fixed r , $1 \leq r \leq n-1$, consider the sub-domain $\{(a, \theta) : a \geq 1/2, x_{(r)} \leq \theta \leq x_{(r+1)}\}$. On this sub-domain

$$\begin{aligned} \frac{\partial \log L}{\partial \theta} &= -ar + b(n-r) \\ &= br \left\{ \frac{n-r}{r} - \frac{a}{b} \right\} \\ &= \frac{a}{2a-1} r \left\{ \frac{n-r}{r} - (2a-1) \right\} \\ &= \frac{2ar}{2a-1} \left\{ \frac{n}{2r} - a \right\}, \end{aligned}$$

which is positive or negative according as $a < n/2r$ or $a > n/2r$.

Thus, for $a < n/2r$ ($a > n/2r$), L is increasing (decreasing) function of θ for $x_{(r)} \leq \theta \leq x_{(r+1)}$ and for $a = n/2r$, L is constant for $x_{(r)} \leq \theta \leq x_{(r+1)}$. Thus to find m.l.e.s of a and θ one can confine to the domain

$$\{(a, \theta) : a \geq 1/2, \theta \in \{x_{(1)}, \dots, x_{(n)}\}\}.$$

Consider

$$L(a, x_{(r)}) = \frac{1}{2^n} \exp \left\{ a \sum_{i=1}^{r-1} (x_{(i)} - x_{(r)}) - b \sum_{i=r}^n (x_{(i)} - x_{(r)}) \right\}$$

$$\text{Now, } \frac{\partial \log L(a, x_{(r)})}{\partial a} = 0$$

$$\Rightarrow \left\{ \sum_{i=1}^{r-1} (x_{(i)} - x_{(r)}) + \frac{b^2}{a^2} \sum_{i=r}^n (x_{(i)} - x_{(r)}) \right\} = 0$$

$$\Rightarrow \frac{a^2}{b^2} = \frac{\left\{ \sum_{i=r}^n (x_{(i)} - x_{(r)}) \right\}}{\left\{ \sum_{i=1}^{r-1} (x_{(i)} - x_{(r)}) \right\}}$$

$$\hat{a}_r = \frac{1}{2} \left(\left\{ \frac{\sum_{i=r}^n (x_{(i)} - x_{(r)})}{\sum_{i=1}^{r-1} (x_{(i)} - x_{(r)})} \right\}^{\frac{1}{2}} + 1 \right)$$

More over we have,

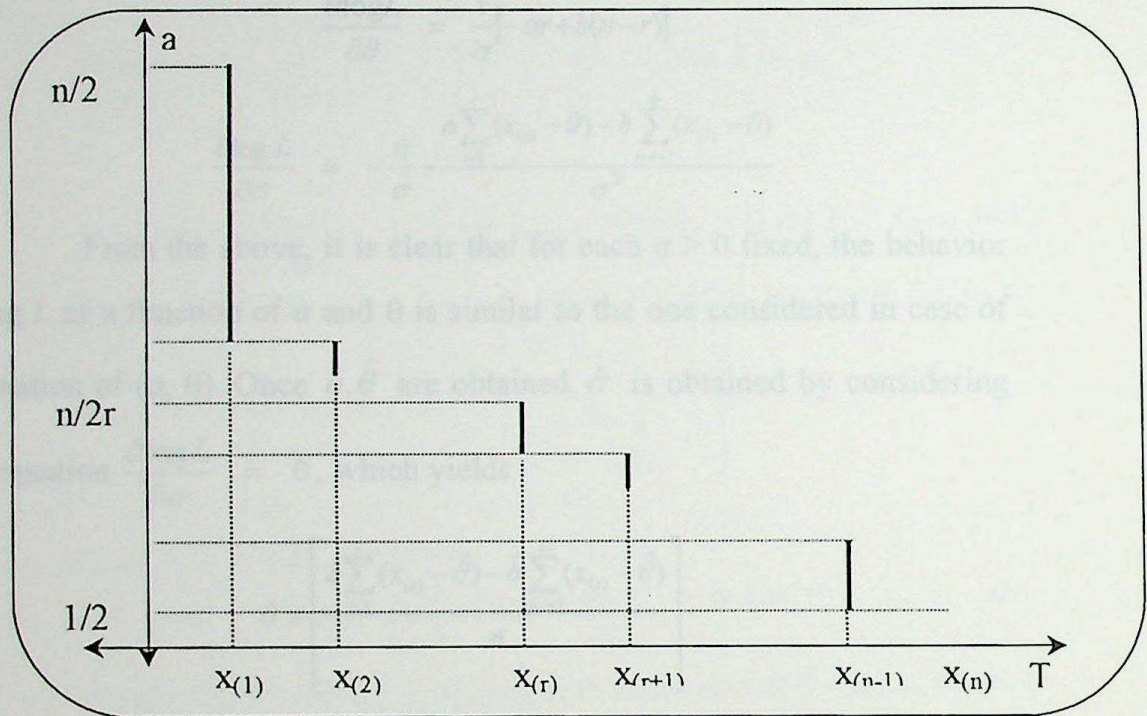
$$L(a, x_{(r)}+) = \frac{1}{2^n} \exp \left\{ a \sum_{i=1}^r (x_{(i)} - x_{(r)}) - b \sum_{i=r+1}^n (x_{(i)} - x_{(r)}) \right\} \text{ and}$$

$$L(a, x_{(r)}-) = \frac{1}{2^n} \exp \left\{ a \sum_{i=1}^{r-1} (x_{(i)} - x_{(r)}) - b \sum_{i=r}^n (x_{(i)} - x_{(r)}) \right\}.$$

Thus, for a fixed $L(a, \theta)$ is a continuous function of θ for $-\infty < \theta < \infty$.

Hence, for each a fixed with $a < n/2r$ ($a > n/2r$) the function $L(a, \theta)$ is increasing (decreasing) in θ for $\theta < x_{(r)}$ ($\theta > x_{(r)}$). As such, the domain for the search of MLE's is $\{(a, x_{(r)}) : a \in [n/2r, n/2(r-1)], r = 1, 2, \dots, n\}$. Following method can be used to find mle of (a, θ) :

The domain for the search of MLE's



Step 1 : Compute $a_r = \frac{1}{2} \left(\left\{ \frac{\sum_{i=r}^n (x_{(i)} - x_{(r)})}{\sum_{i=1}^{r-1} (x_{(i)} - x_{(r)})} \right\}^{\frac{1}{2}} + 1 \right).$

Step 2 : Define $A = \left\{ r : a_r \in \left[\frac{n}{2r}, \frac{n}{2(r-1)} \right] \right\}.$

Step 3 : Compute $h_r = \left\{ a_r \sum_{i=1}^r (x_{(i)} - x_{(r)}) - b_r \sum_{i=r+1}^n (x_{(i)} - x_{(r)}) \right\}$ for $r \in A.$

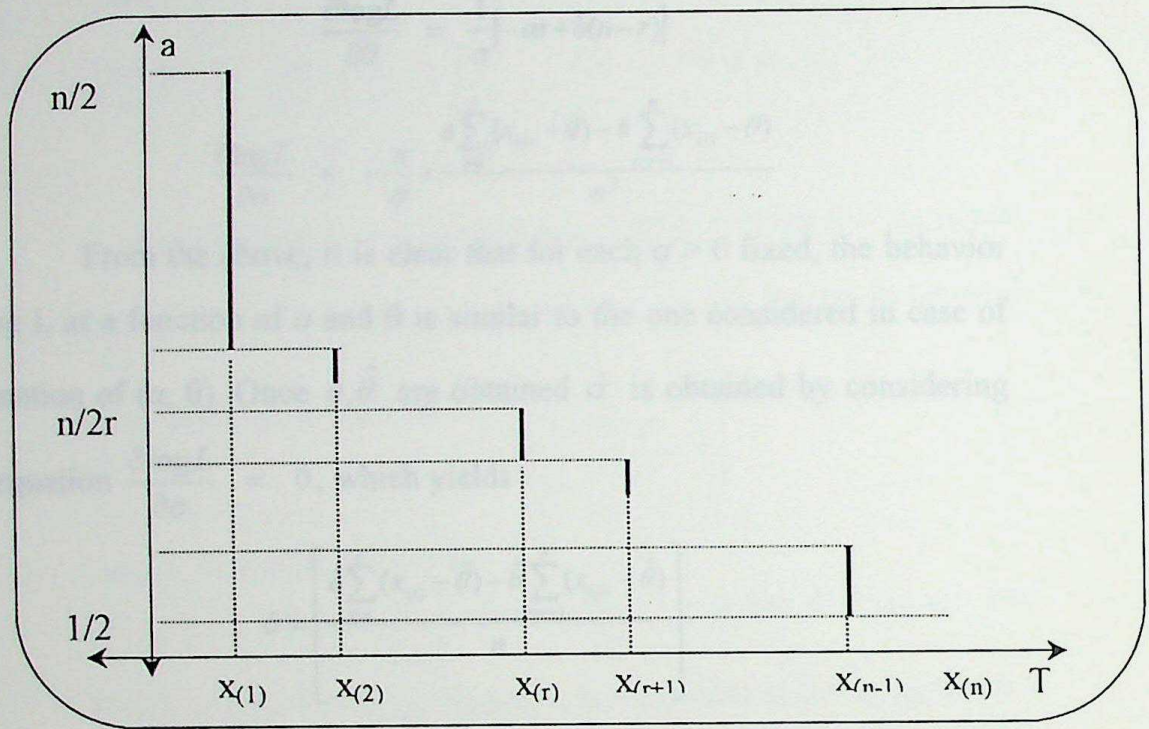
Step 4 : Find $r^* \ni h_{r^*} \geq h_r : r \in A.$

Step 5 : $(\hat{a}, \hat{\theta}) = (a_{r^*}, x_{(r^*)}).$

(II) Estimation of (a, θ, σ) : For $x_{(r)} < \theta \leq x_{(r+1)},$

$$L(a, \theta, \sigma) = \frac{1}{(2\sigma)^n} \exp \left\{ \frac{a}{\sigma} \sum_{i=1}^r (x_{(i)} - \theta) - \frac{b}{\sigma} \sum_{i=r+1}^n (x_{(i)} - \theta) \right\}$$

The domain for the search of MLE's



Step 1 : Compute $a_r = \frac{1}{2} \left(\left(\frac{\sum_{i=r}^n (x_{(i)} - x_{(r)})}{\sum_{i=1}^{r-1} (x_{(i)} - x_{(r)})} \right)^{1/2} + 1 \right).$

Step 2 : Define $A = \left\{ r : a_r \in \left[\frac{n}{2r}, \frac{n}{2(r-1)} \right] \right\}.$

Step 3 : Compute $h_r = \left\{ a_r \sum_{i=1}^r (x_{(i)} - x_{(r)}) - b_r \sum_{i=r+1}^n (x_{(i)} - x_{(r)}) \right\}$ for $r \in A.$

Step 4 : Find $r^* \ni h_{r^*} \geq h_r : r \in A.$

Step 5 : $(\hat{a}, \hat{\theta}) = (a_{r^*}, x_{(r^*)}).$

(II) Estimation of (a, θ, σ) : For $x_{(r)} < \theta \leq x_{(r+1)},$

$$L(a, \theta, \sigma) = \frac{1}{(2\sigma)^n} \exp \left\{ \frac{a}{\sigma} \sum_{i=1}^r (x_{(i)} - \theta) - \frac{b}{\sigma} \sum_{i=r+1}^n (x_{(i)} - \theta) \right\}$$

$$\text{and } \frac{\partial \log L}{\partial a} = \frac{1}{\sigma} \left\{ \sum_{i=1}^r (x_{(i)} - \theta) + \frac{b^2}{a^2} \sum_{i=r+1}^n (x_{(i)} - \theta) \right\}$$

$$\frac{\partial \log L}{\partial \theta} = \frac{1}{\sigma} [-ar + b(n-r)]$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} - \frac{a \sum_{i=1}^r (x_{(i)} - \theta) - b \sum_{i=r+1}^n (x_{(i)} - \theta)}{\sigma^2}$$

From the above, it is clear that for each $\sigma > 0$ fixed, the behavior of $\log L$ as a function of a and θ is similar to the one considered in case of estimation of (a, θ) . Once $\hat{a}, \hat{\theta}$ are obtained $\hat{\sigma}$ is obtained by considering the equation $\frac{\partial \log L}{\partial \sigma} = 0$, which yields

$$\hat{\sigma} = \left[\frac{\hat{a} \sum_{i=1}^r (x_{(i)} - \hat{\theta}) - \hat{b} \sum_{i=r+1}^n (x_{(i)} - \hat{\theta})}{n} \right]$$

(III) M.L.E. of parameter when other two are known

(i) Estimation of a :

In this case,

$$\begin{aligned} \frac{\partial \log L}{\partial a} &= \frac{1}{\sigma} \left\{ \sum_{i=1}^r (x_{(i)} - \theta) + \frac{b^2}{a^2} \sum_{i=r+1}^n (x_{(i)} - \theta) \right\} = 0 \\ \Rightarrow \hat{a} &= \frac{1}{2} \left(\left[\frac{\sum_{i=r+1}^n (x_{(i)} - \theta)}{\sum_{i=1}^r (x_{(i)} - \theta)} \right]^{\frac{1}{2}} + 1 \right) \end{aligned} \quad \text{----- (4)}$$

(ii) Estimation of σ :

$$\begin{aligned} \frac{\partial \log L}{\partial \sigma} &= -\frac{n}{\sigma} - \frac{a \sum_{i=1}^r (x_{(i)} - \theta) - b \sum_{i=r+1}^n (x_{(i)} - \theta)}{\sigma^2} \\ \Rightarrow \hat{\sigma} &= \left[\frac{a \sum_{i=1}^r (x_{(i)} - \theta) - b \sum_{i=r+1}^n (x_{(i)} - \theta)}{n} \right] \end{aligned} \quad \text{----- (5)}$$

(iii) Estimation of θ :

Case I : $a = n/2k$, for some $k = 1, 2, \dots, n$

$$\begin{aligned}\frac{\partial \log L}{\partial \theta} &= \frac{2ar}{2a-1} \left(\frac{n}{2r} - a \right), \quad \text{for } x_{(r)} < \theta \leq x_{(r+1)} \\ &= \frac{2a^2}{2a-1} \left(\frac{n}{2a} - r \right)\end{aligned}$$

Hence, $\frac{\partial \log L}{\partial \theta} > 0$ for $\theta \leq x_{(r)}$, $\frac{\partial \log L}{\partial \theta} < 0$ for $\theta > x_{(r+1)}$ and $\frac{\partial \log L}{\partial \theta} = 0$ for $\theta \in (x_{(r)}, x_{(r+1)}]$, where $r = n/2a$. Hence, m.l.e. of θ

is any value in the interval $\left[x_{\left(\frac{n}{2a}\right)}, x_{\left(\frac{n}{2a}+1\right)} \right]$.

Case II : $a \neq n/2k$, for any $k = 1, 2, \dots, n$

Let $a \in \left(\frac{n}{2(r+1)}, \frac{n}{2r} \right]$, then $\frac{\partial \log L}{\partial \theta} > 0$ for $\theta < x_{(r+1)}$ and

$\frac{\partial \log L}{\partial \theta} < 0$ for $\theta > x_{(r+1)}$. Hence, L is maximum at $\theta = x_{(r+1)}$ where r

is $\exists a \in \left(\frac{n}{2(r+1)}, \frac{n}{2r} \right]$.

(IV) M.L.E.s of parameters when other one is known

(i) Estimation of (a, σ) :

When θ is known, $\hat{a}$ and $\hat{\sigma}$ are respectively given by (4) and (5).

(ii) Estimation of (θ, a) :

When σ is known, the estimators $\hat{a}, \hat{\theta}$ are as given in case (I) of section 3.3.1.

(iii) Estimation of (θ, σ) :

When a is known $\hat{\theta}$ is same as the one obtained when both a and σ are known. Further, $\hat{\sigma}$ is given by

$$\hat{\sigma} = \left[\frac{a \sum_{i=1}^r (x_{(i)} - \hat{\theta}) - b \sum_{i=r+1}^n (x_{(i)} - \hat{\theta})}{n} \right].$$

4. Drifted Standard Normal Distribution :

The pdf corresponding to drifted distribution obtained from SNV is given by

$$g(x, a) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-a^2 x^2 / 2} & \text{for } x < 0 \\ \frac{1}{\sqrt{2\pi}} e^{-b^2 x^2 / 2} & \text{for } x \geq 0. \end{cases}$$

Remarks

1. This distribution does not belong to one parameter exponential family. However, it belongs to Cramer family.
2. The drifted distribution obtained from $N(0, \sigma^2)$ belongs to two parameter exponential family.

4.1 Some Distributional Properties :

$$(i) \quad \text{Mean} = \frac{1}{\sqrt{2\pi}} 4(1 - \frac{1}{a}).$$

(Since we know that, $\mu_r = \frac{\mu_r}{a^{r+1}} ((2a-1)^{r+1} + (-1)^r)$ and in this case, $\mu^* = \frac{1}{\sqrt{2\pi}}$.)

$$(ii) \quad \text{Variance} = \frac{(4\pi - 8)a^2 - (6\pi - 16)a + (3\pi - 1)}{a^2\pi}.$$

4.2 Estimation of the drift parameter a :

Consider the problem of estimation of the drifted parameter a when the distribution is derived from the standard normal distribution

(I) M.L.E. of a :

Let r of the n random observations from g be negative and remaining $n-r$ be positive. Let $U = \sum_{x_i < 0} x_i^2$ and $V = \sum_{x_i \geq 0} x_i^2$. The loglikelihood function is given by

$$\text{Log } L = \log c - \frac{1}{2} [a^2 U + b^2 V].$$

Case (a) If $r = 0$, i.e. $U = 0$, $\text{Log } L = \log c - \frac{1}{2} [b^2 V]$ which is a decreasing function of b and hence increasing function of a . Hence, mle of a does not exist. However in this case, 'n' can be taken as a modified estimate of a .

Case (b) If $r = n$, i.e. $V = 0$, $\text{Log } L = \log c - \frac{1}{2} [a^2 U]$ which is a decreasing function of a . Hence, mle of a is $\hat{a} = \frac{1}{2}$.

$$\text{Case (c) For } 0 < r < n, \quad \frac{\partial \log L}{\partial a} = 0$$

$$\Rightarrow [2aU + 2bV \frac{db}{da}] = 0.$$

The unique solution of the above equation is $\hat{a} = \frac{1}{2} \left[1 + \left(\frac{V}{U} \right)^{1/3} \right]$.

Hence, for $0 < r < n$, mle of a is $\hat{a} = \frac{1}{2} \left[1 + \left(\frac{V}{U} \right)^{1/3} \right]$.

II. The Fisher information contained in a single observation :

Differentiating the logarithm of pdf w.r.t. a we get

$$\frac{\partial \log g}{\partial a} = \begin{cases} -ax^2 & \text{for } x < 0 \\ \frac{a}{(2a-1)^3} x^2 & \text{for } x > 0 \end{cases}$$

$$E \left[\frac{\partial \log g}{\partial a} \right]^2 = \int_{-\infty}^0 \frac{a^2 x^4}{\sqrt{2\pi}} e^{-a^2 x^2/2} dx + \int_0^{\infty} \frac{a^2 x^4}{(2a-1)^6 \sqrt{2\pi}} e^{-b^2 x^2/2} dx$$

$$\text{i.e. } I(a) = \int_0^{\infty} \frac{(2a^2 u)^{3/2}}{\sqrt{2\pi}} e^{-u} du + \int_0^{\infty} \frac{(2a^2 v)^{3/2}}{(2a-1)^4 \sqrt{2\pi}} e^{-v} dv$$

$$\text{where } u = \frac{a^2 x^2}{2} \text{ and } v = \frac{b^2 x^2}{2}$$

$$= \frac{2a^3}{\sqrt{\pi}} \left| \left(\frac{5}{2} \right) \right| + \frac{2b^3}{(2a-1)^4 \sqrt{\pi}} \left| \left(\frac{5}{2} \right) \right|$$

$$I(a) = \frac{3}{2}a^3 \left[1 + \frac{1}{(2a-1)^7} \right].$$

4.3 Drifted Normal Distribution with location and scale parameter :

By introducing location and scale parameters in a drifted distribution obtained from standard normal distribution, we get the following pdf

$$g(x, \theta, \sigma, a) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{a^2(x-\theta)^2}{2\sigma^2}\right\} & \text{for } x < \theta \\ \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{b^2(x-\theta)^2}{2\sigma^2}\right\} & \text{for } x \geq \theta. \end{cases}$$

This is essentially a reparameterized version of binormal (two piece normal) distribution considered by John (1982). He has obtained the M.L.E.'s of the parameters.

MLE of estimator (s) when other one or two parameters are known can be found.

(i) MLE of one parameter when the other two are known

Estimation of a :

By considering the variables $(X_i - \theta)/\sigma$ it reduces to the earlier case and we get

$$\hat{a} = \frac{1}{2} \left[1 + \left(\frac{\sum_2 (x_{(j)} - \theta)^2}{\sum_1 (\theta - x_{(i)})^2} \right)^{\frac{1}{3}} \right],$$

where $\sum_1$ and $\sum_2$ are the sums taken over the observations smaller than θ and larger than θ respectively.

Estimation of θ :

The MLE of θ is the solution to equation

$$\frac{a^2}{b^2} = \frac{\sum_2 (x_{(j)} - \theta)}{\sum_1 (\theta - x_{(i)})}, \quad \text{where } b^{-1} = 2 - a^{-1}.$$

Estimation of σ :

The MLE of σ is

$$\hat{\sigma} = \left[\frac{a^2 \sum_1 (\theta - x_{(i)})^2 + b^2 \sum_2 (x_{(j)} - \theta)^2}{n} \right]^{1/2}.$$

(ii) MLE of two parameters when the other one is known

Estimation of θ and σ :

Nabar and Barpande (2001) have obtained the MLE of θ and σ when a is known (in their notation it is k).

Estimation of a and σ :

Here, MLE's are given by

$$\hat{a} = \frac{1}{2} \left[1 + \left(\frac{\sum_2 (x_{(j)} - \theta)^2}{\sum_1 (\theta - x_{(i)})^2} \right)^{1/3} \right] \quad \text{and} \quad \hat{\sigma} = \left[\frac{\hat{a}^2 \sum_1 (\theta - x_{(i)})^2 + \hat{b}^2 \sum_2 (x_{(j)} - \theta)^2}{n} \right]^{1/2}.$$

Estimation of θ and a :

$$L(\theta, a) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2} \left[a^2 \sum_1 (\theta - x_i)^2 + b^2 \sum_2 (x_i - \theta)^2 \right] \right\}$$

We need to minimize

$$h(a, \theta) = \begin{cases} b^2 \sum_2 (x_i - \theta)^2 & \text{if } \theta \leq x_{(1)} \\ a^2 \sum_1 (\theta - x_i)^2 + b^2 \sum_2 (x_i - \theta)^2 & \text{if } x_{(1)} < \theta < x_{(n)} \\ a^2 \sum_1 (\theta - x_i)^2 & \text{if } \theta \geq x_{(n)} \end{cases}$$

Hence, for each a fixed $h(a, \theta) \downarrow$ in θ for $\theta \leq x_{(1)}$ and $h(a, \theta) \uparrow$ in θ for $\theta \geq x_{(n)}$. Thus, to minimize $h(a, \theta)$ it is enough to confine to the values of θ between $x_{(1)}$ and $x_{(n)}$.

$$\begin{aligned} \frac{\partial \log L}{\partial a} &= 0 \\ \Rightarrow \frac{a^3}{b^3} &= \frac{\sum_1^2 (x_{(j)} - \theta)^2}{\sum_1^2 (\theta - x_{(i)})^2} \\ \frac{\partial \log L}{\partial \theta} &= 0 \\ \Rightarrow \frac{a^2}{b^2} &= \frac{\sum_1^2 (x_{(j)} - \theta)}{\sum_1^2 (\theta - x_{(i)})} \end{aligned}$$

Hence, from the above two equations, θ is such that

$$\left(\frac{\sum_1^2 (x_{(j)} - \theta)^2}{\sum_1^2 (\theta - x_{(i)})^2} \right)^{\frac{1}{3}} = \left(\frac{\sum_1^2 (x_{(j)} - \theta)}{\sum_1^2 (\theta - x_{(i)})} \right)^{\frac{1}{2}}$$

Let $g_1(\theta) = \left(\frac{\sum_1^2 (x_{(j)} - \theta)^2}{\sum_1^2 (\theta - x_{(i)})^2} \right)^{\frac{1}{3}}$ and $g_2(\theta) = \left(\frac{\sum_1^2 (x_{(j)} - \theta)}{\sum_1^2 (\theta - x_{(i)})} \right)^{\frac{1}{2}}$

Hence, we have to find θ such that $g_1(\theta) = g_2(\theta)$. Note that both $g_1(\theta)$ & $g_2(\theta)$ are decreasing functions of θ . Consider $(x_{(1)}, x_{(n)})$ as the domain for θ . We observe that $g_2(\theta) > g_1(\theta)$ in the neighborhood of $x_{(1)}$ and $g_2(\theta) < g_1(\theta)$ in the neighborhood of $x_{(n)}$. Further as g_1 and g_2 are continuous, there must be at least one solution for $g_1(\theta) = g_2(\theta)$ in between $x_{(1)}$ and $x_{(n)}$. If multiple solutions occur, they must be odd in number. Corresponding to each θ^* , a solution of $g_1(\theta) = g_2(\theta)$, we obtain a^* , the value of a , and the value of $h(\theta^*, a^*)$. The pair which minimizes h is the mle for (θ, a) .

Program in C language enclosed in Appendix 3.1 has been developed to compute the mle's in all the above cases from the data simulated from drifted standard normal distribution.

4.4 Simulation study of MLE's :

To study small, moderate and large sample behavior of m.l.e., simulation study was carried out by taking samples of sizes $N = 10, 20, 50, 100, 200, 300$ and generating M samples each time.

The averages of M estimators and their variances are given in the following table :

A Simulation Study For The Performance of M.l.e.'s

$$M = 100, a = 1, \theta = 0, \text{ and } \sigma^2 = 1$$

N	θ	a	σ^2	$V(\theta)$	$V(a)$	$V(\sigma^2)$
10	0.1334	1.2815	1.0229	1.1839	1.3403	0.3338
20	0.0234	1.4345	0.9812	0.6319	2.6119	0.0993
50	-0.0032	1.0811	0.9412	0.1512	0.1912	0.0525
100	0.0618	0.9823	0.9585	0.0589	0.0202	0.0176
200	0.0152	1.0032	0.9704	0.0362	0.0132	0.0081
300	0.0046	1.0085	0.9898	0.0213	0.0083	0.0073

Parameters	Sample Size				When θ known		When a known	
		$\hat{\theta}$	$\hat{a}$	$\hat{\sigma}^2$	$\hat{a}$	$\hat{\sigma}^2$	$\hat{\theta}$	$\hat{\sigma}^2$
$a = 0.8$ $\theta = 5$ $\sigma^2 = 1$	10	6.7307	0.5655	1.1496	1.2358	0.8711	5.614	0.9669
	20	4.4867	1.0039	1.7868	0.8236	1.7954	5.0972	1.7959
	50	4.1887	1.3006	1.3024	0.8802	1.4105	5.0743	1.4651
	100	4.6178	1.0553	1.3942	0.8972	1.4227	5.1974	1.4694
	200	4.6559	1.0023	1.4244	0.8704	1.4472	5.154	1.473
	300	4.3618	1.1125	1.803	0.8759	1.8889	5.1274	1.9469
$a = 1$ $\theta = 5$ $\sigma^2 = 1$	10	5.2759	1.3442	0.4213	1.7899	0.4479	5.5127	0.4492
	20	4.9193	1.33	0.8764	1.2516	0.8771	5.3016	0.8937
	50	5.4029	0.7266	0.7048	0.9041	0.7159	4.8769	0.7241
	100	5.3563	0.8537	1.0245	1.0122	1.0419	5.0801	1.0365
	200	5.1543	0.953	1.0669	1.0304	1.0681	5.0679	1.0667
	300	4.8449	1.1037	0.9179	1.0061	0.9225	4.9872	0.9224
$a = 1.2$ $\theta = 5$ $\sigma^2 = 1$	10	5.0293	0.7049	0.4372	0.7172	0.437	4.3711	0.471
	20	5.4744	0.8132	0.5676	1.0625	0.6265	5.0306	0.6484
	50	5.1791	0.8705	0.7457	0.97	0.747	4.7161	0.7539
	100	5.3179	0.8933	0.6947	1.0902	0.7073	4.9296	0.7178
	200	5.1413	0.9669	0.8075	1.0546	0.8093	4.8395	0.8166
	300	5.1541	1.0193	0.7149	1.1296	0.7187	4.954	0.7226

5. **Case studies :** From the point of convenience in grocery shops articles to be sold are packed and kept ready for the customers. To maintain the goodwill of the customers, owners might be taking additional precautions for packages not to be under weighted. This would possibly affect the behavior of X , the weight of packages in the following manner. The mean of the variable is more than the target value and the errors are symmetrically distributed and / or the distribution is positively skewed. From the point of view of estimating and testing the validity of the model we collected data from a super market. We thank the management of the market for permitting us to collect the required data. We collected the data regarding weights of packages of the commodities *Shabu*, *Jaypatri* and a branded tea package.

Quiz competition tests for first year (F.Y.) B.Sc. and second year (S.Y.) B.Sc. (Statistics) students from the colleges affiliated to Shivaji University are conducted. The frequency distributions of the marks obtained out of 100 by these students for the examinations conducted in February, 2004 have been considered.

These data sets have been used for testing the validity of the models.

A. Drifted normal model :

Data Set I : *Shabu*

Packages labeled 500 grams were considered. The average weight of the polythene bag used for packing was 2.85 grams.

Frequency Table :-

Values	Frequencies
501	2
501.5	1
502	2
502.5	10
503	13
503.5	22
504	16
504.5	14
505	12
505.5	2
506	2
506.5	1
Total	97

Relevant Statistics :-

$u =$	47.07689	$\text{Theta} =$	503.806
$v =$	55.1498	$a =$	0.9743
$w =$	37.618	$\text{sigma_sq} =$	1.05169
$z =$	-41.8	$b =$	1.02708
Mean =	503.7629	S.D =	1.031

Data Set II : Aspirin

Packages labeled 20 grams were considered. The average weight of the aspirin bag used for packing was 1.1 grams.

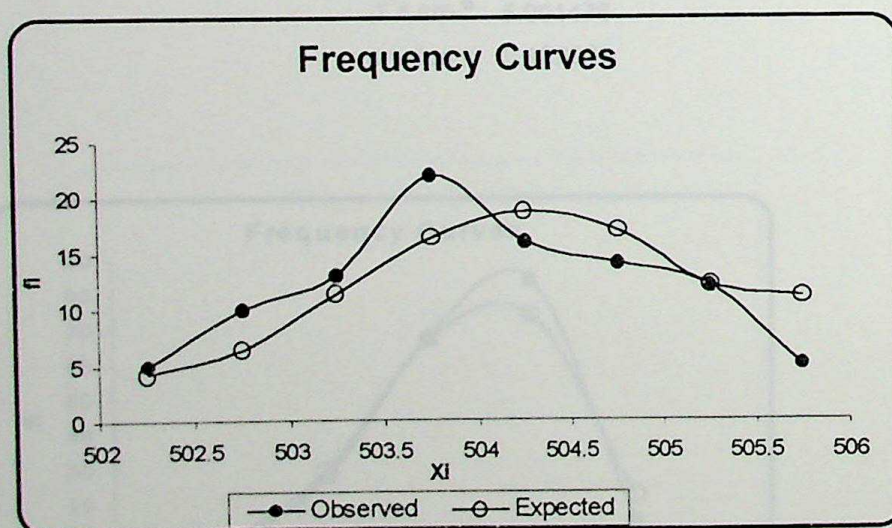
TABLE FOR GOODNESS OF FIT

Class Intervals	O _i	P _i	E _i	(O _i - E _i) ² /E _i
Upto 502.25	5	0.0715017	6.9356682	0.540223563
502.25-502.75	10	0.0905283	8.7812458	0.169151607
502.75-503.25	13	0.1445151	14.017963	0.073922847
503.25-503.75	22	0.1848664	17.932043	0.922832666
503.75-504.25	16	0.1889699	18.330079	0.296194579
504.25-504.75	14	0.151944	14.738567	0.03701051
504.75-505.25	12	0.0955682	9.2701111	0.803905483
505.25 onwards	5	0.0721064	6.9943226	0.568650134
	97	1	97	3.411891389

$$\chi^2_{5,0.05} = 11.07048257$$

$$p\text{-value} = 0.636759488$$

Here, Cal. $\chi^2_5 = 3.4119$, Tab. $\chi^2_{5,0.05} = 11.07$ and p-value is 0.6368.



Data Set II : Jaypatri

Packages labeled 20 grams were considered. The average weight of the polythene bag used for packing was 1.1 grams.

Frequency Table :-

Values	Frequencies
19.5	2
20	4
20.5	27
21	68
21.5	84
22	12
22.5	3
Total	200

Relevant Statistics :-

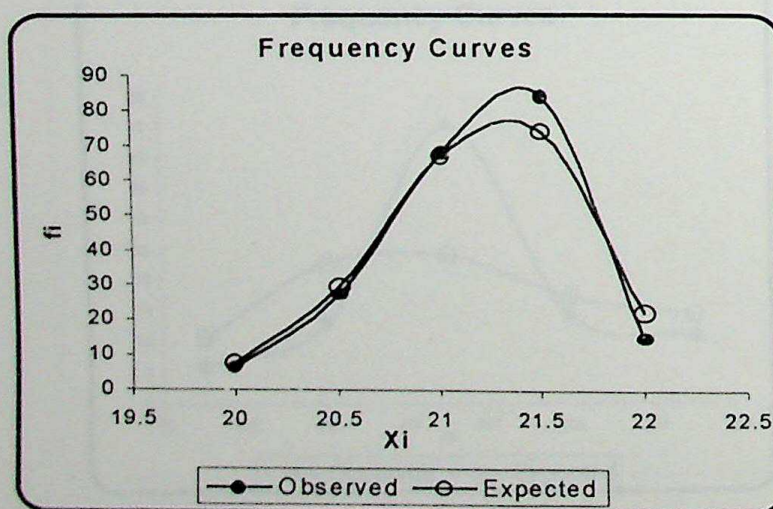
$u = 9.57792$	$\text{Theta} = 21.38018$
$v = 45.43607$	$a = 0.797573$
$w = 20.86178$	$\text{sigma_sq} = 0.230522$
$z = -58.8986$	$b = 1.34013$
$\text{Mean} = 21.19$	$\text{S.D.} = 0.49$

TABLE FOR GOODNESS OF FIT

Class intervals	O _i	P _i	E _i	(O _i - E _i) ² /E _i
upto 20.25	6	0.03790197	7.580394	0.329487366
20.25-20.75	27	0.14714179	29.42836	0.20038228
20.75-21.25	68	0.33452424	66.90485	0.017926317
21.25-21.75	84	0.36776971	73.55394	1.483538789
21.75 onwards	15	0.11266229	22.53246	2.518053225
Total	200	1	200	4.549387977

$$\chi^2_{5, 0.05} = 5.991476$$

$$p\text{-value} = 0.102828$$



Data Set III : A branded tea package

Packages labeled 20 grams were considered. The average weight of the polythene bag used for packing was 1.1 grams.

Frequency
Table :-

Values	Freq.
252	1
252.5	1
253	3
253.5	1
254	3
254.5	5
255	5
255.5	9
256	17
256.5	19
257	8
257.5	2
258	4
258.5	2
259	0
259.5	1
260	1
260.5	0
261	0
261.5	2
262	1
262.5	1
263	3
Total	89

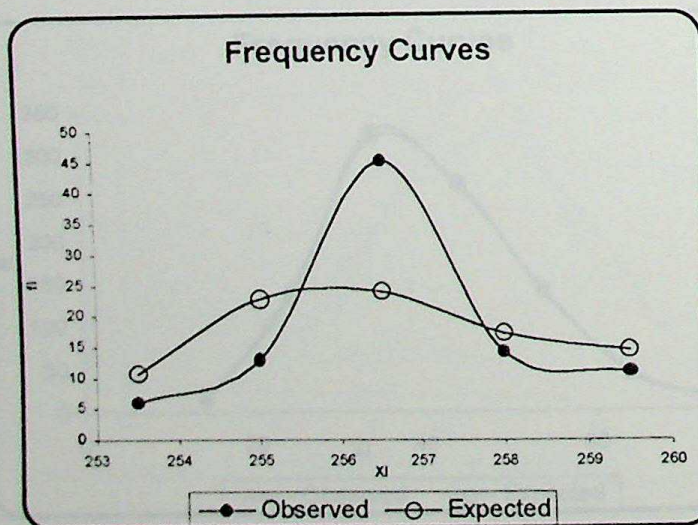
Relevant
Statistics :-

$u = 447.8609$ $v = 60.1814$ $w = 112.6278$ $z = 29.5479$ Mean = 256.488	Theta = 255.555 $a = 1.4762$ $b = 0.7561$ Sigma $sq = 4.3503$ S.D. = 2.2118
--	---

Class intervals	O _i	P _i	E _i	χ^2
Upto 254.25	6	0.1204	10.7187	2.077289
254.25-255.75	13	0.2555	22.738	4.170476
255.75-257.25	45	0.2677	23.8210	18.82982
257.25-258.75	14	0.1932	17.1954	0.593808
258.75 onwards	11	0.1632	14.5269	0.856254
Total	89	1	89	26.52765

$$\chi^2_{5, 0.05} = 5.9914764$$

$$p\text{-value} = 1.736E-06$$



Data Set IV : F.Y. B. Sc. Quiz Marks

Freq. Table :

Relevant Statistics :-

Values	Freq.	Values	Freq.
8	1	44	52
10	1	46	44
12	2	48	34
14	1	50	38
16	2	52	37
18	7	54	20
20	6	56	23
22	19	58	22
24	26	60	17
26	42	62	8
28	37	64	5
30	62	66	7
32	67	68	4
34	68	70	3
36	56	72	5
38	70	74	3
40	64	76	3
42	66		
Total		922	

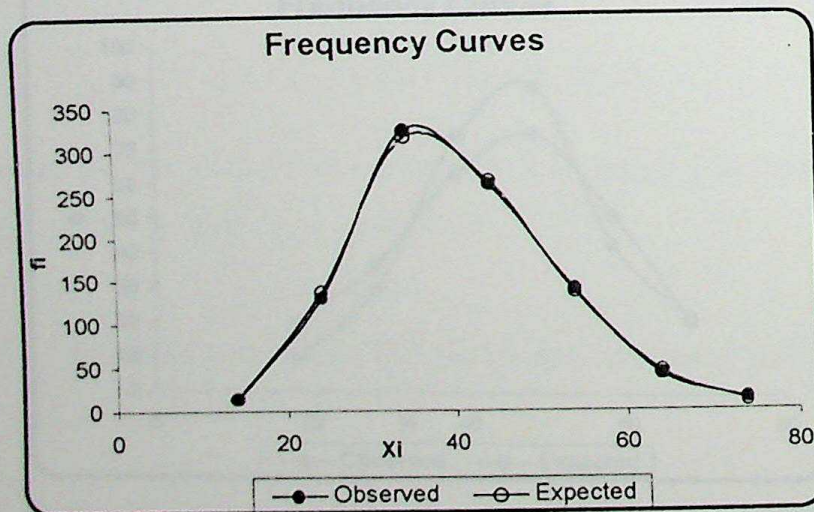
$u = 129032.2$	Theta = 34.4074
$v = 18371.87$	$\alpha = 1.4576$
$w = 7019.301$	$b = 0.7611$
$z = -1913.92$	Sigma-sq = 123.4
Mean = 39.94469	S.D. = 11.3734

Fitting to drifted normal :-

Class Intervals	O _i	P _i	E _i	ChiSq.
upto 19	14	0.0148257	13.6693	0.008
19 to 29	130	0.1491549	137.520	0.4113
29 to 39	323	0.3413206	314.697	0.2190
39 to 49	260	0.2861793	263.857	0.0564
49 to 59	140	0.1480815	136.531	0.0881
59 to 69	41	0.0487553	44.9523	0.3475
69 onwards	14	0.0116827	10.7715	0.9677
Total	922	1	922	2.0980

$$\chi^2_{3, 0.05} = 7.81473$$

$$p\text{-value} = 0.55231$$



Data Set V : S. Y. B. Sc. Quiz Marks

Frequency
Table :

Values	Freq.	Values	Freq.
0	10	54	16
24	3	56	8
26	4	58	7
28	6	60	9
30	7	62	3
32	10	64	4
34	15	66	4
36	15	68	8
38	19	70	1
40	11	72	1
42	14	74	1
44	22	76	1
46	24	78	0
48	16	80	0
50	14	82	2
52	13	Total	268

Relevant Statistics :-

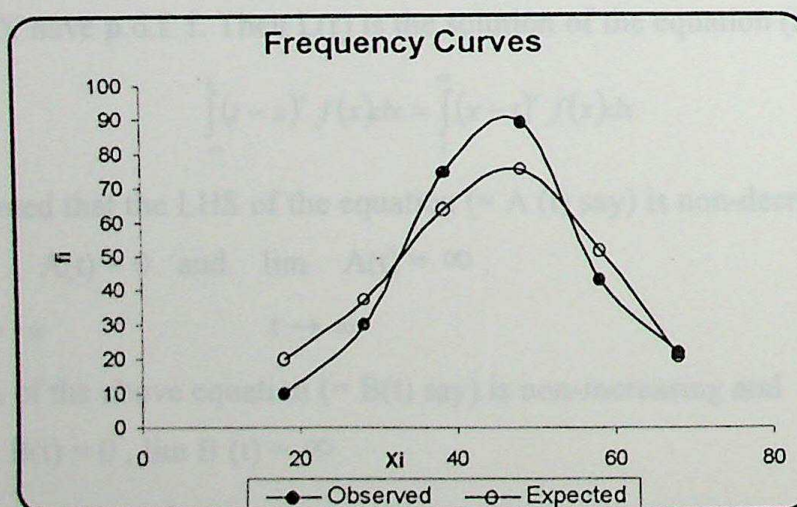
$u = 14677.06$	$\text{Theta} = 47.792$
$v = 41806.96$	$a = 0.85272$
$w = 946.464$	$b = 1.20877$
$z = -1901.72$	$\text{Sigma-sq} = 193.449$
$\text{Mean} = 44.22761$	$\text{S.D.} = 14.0996$

Fitting to drifted normal :-

Class Intervals	O _i	P _i	E _i	ChiSq
upto 23	10	0.0753574	20.1958	5.14731
23 to 33	30	0.1383515	37.0782	1.35122
33 to 43	74	0.2371507	63.5564	1.7161
43 to 53	89	0.2799317	75.0217	2.60449
53 to 63	43	0.1921602	51.4989	1.40259
63 onwards	22	0.0770486	20.649	0.08839
	268	1	268	12.3101

$$\chi^2_{3, 0.05} = 7.814725$$

$$p\text{-value} = 0.006393$$



Drifted normal model gives good fit for the data sets I, II and IV while it does not give good fit for data sets III and V (at 5% level of significance).

B. Drifted Laplace model :

For the above data sets goodness of fit of drifted Laplace (three-parameter) model by using MLE's was attempted. The program in C language to compute MLE's of given data is enclosed in Appendix 3.2. The drifted Laplace model is not giving a good fit.

We note that in case of drifted Laplace model, MLE of θ is an order statistic, which is being much affected by grouping. In the above sets of data, exact weights were not available. Observations are integers or mid-values of integers and the range of observations is small. These facts might have affected the goodness of fit for drifted Laplace model.

6. Some New Measure of Location

Through this study, we are motivated to introduce a wide class of location measures. We refer them as ' r^{th} exponentiated measure of Location', for $r \geq 0$ and denote them by $L(r)$.

Definition of $L(r)$:

Let X have p.d.f. f . Then $L(r)$ is the solution of the equation (in t)

$$\int_{-\infty}^t (t-x)^r f(x) dx = \int_t^{\infty} (x-t)^r f(x) dx.$$

It is to be noted that the LHS of the equation (= $A(t)$ say) is non-decreasing and

$$\lim_{t \rightarrow -\infty} A(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} A(t) = \infty.$$

While RHS of the above equation (= $B(t)$ say) is non-increasing and

$$\lim_{t \rightarrow \infty} B(t) = 0, \quad \lim_{t \rightarrow -\infty} B(t) = \infty$$

Hence, if X has p.d.f f then there exist at list one solution of the equation.

We note that

- (i) $L(0) = \text{Median}$, $L(1) = E(X)$.
- (ii) If F is symmetric about θ then $L(r) = \theta \quad \forall r \geq 0$.

If F is not continuous $L(r)$ is any value such that $l \leq L(r) \leq u$, where

$$l = \sup \{ t : A(t) \leq B(t) \}$$

$$u = \inf \{ t : B(t) \leq A(t) \}.$$

By using this definition sample r^{th} exponentiated measure of location can be defined based on F_n , the sample empirical distribution function.

Appendix – 3.1

/* To find MLE's of θ , S^2 and a in four cases for data simulated from drifted normal distribution */

```
#include <stdio.h>
#include <conio.h>
#include <math.h>
#include <stdlib.h>
#include <time.h>
void main()
{
    int i, n, m, k1, j, tn, l, sign[100];
    double L[100];
    float o, p, q, a, b, s, theta, x[1000], temp, r, rt[1000], st[1000];
    float th, t[5000], ltheta[100], la[100], lb[100], sgs[100];
    float u, v, w, z, it[100], ft[100], c, sum1, sum2;
    float sl, al, th1, bl, a2, b2, s2, th3, s3, it3, ft3, d;
    FILE *fp;
    clrscr();
    printf ("Enter sample size :- ");
    scanf ("%d", &n);
    printf ("Enter values of a, Theta, Sigma-Sq. :- ");
    scanf ("%f\n%f\n%f", &a, &theta, &s);
    b = a/(2*a-1);
    randomize();
    for (i=0; i<n; i++)
        x[i] = 0;
    for (i=0; i<n; i++)
    {
        p = (float) random(RAND_MAX)/RAND_MAX;
        if ( p < (1/(2*a)) )
        {
            q = a*p;
            if (q < 0.5)
                r = 1-q;
            else
                r = q;
            fp = fopen("invsn.txt", "r");
            for (j=0; j<5000; j++)
            {
                fscanf(fp, "%f%f", &th, &temp);
                if (r <= th)
                {
                    if (q >= 0.5)
                        x[i] = (sqrt(s)/a)*temp + theta;
                    else
                        x[i] = -(sqrt(s)/a)*temp + theta;
                    break;
                }
            }
            fclose(fp);
        }
        else
        {
            q = b*(p-1/(2*a)) + 0.5;
            if (q < 0.5)
                r = 1-q;
            else
                r = q;
            fp = fopen("invsn.txt", "r");
        }
    }
}
```

```

for (j=0; j<5000; j++)
{
    fscanf(fp,"%f%f",&th,&temp);
    if (r <= th)
    {
        if (q >= 0.5)
            x[i] = (sqrt(s)/a)*temp + theta;
        else
            x[i] = -(sqrt(s)/a)*temp + theta;
        break;
    }
}
fclose(fp);
}

for (i=0; i<n-1; i++)
{
    for (j=i; j<n; j++)
    {
        if (x[i] > x[j])
        {
            temp = x[i];
            x[i] = x[j];
            x[j] = temp;
        }
    }
}

/* To find MLE's of Theta, a & Sigma_sq */

/* To find number of crosses between r & s */

i = 0;
t[0] = x[0] + 0.01;
c = (x[n-1] - x[0])/10;
while( t[i] <= x[n-1] )
{
    i++;
    t[i] = t[i-1] + c;
}
tn = i;
t[tn] = x[n-1] - 0.01;
kl=0;
sign[0]=1;
for (j=0; j<=tn; j++)
{
    w = 0.0;
    z = 0.0;
    u = 0.0;
    v = 0.0;
    for (i=0; i<n; i++)
    {
        if (x[i] > t[j])
        {
            w = w + fabs(x[i] - t[j]);
            u = u + pow((x[i] - t[j]), 2);
        }
        else
        {
            z = z + fabs(x[i] - t[j]);
            v = v + pow((x[i] - t[j]), 2);
        }
    }
}

```

```

    }
}
rt[j] = pow((w/z), (1.0/2));
st[j] = pow((u/v), (1.0/3));
if (st[j] - rt[j] <= 0)
    sign[j+1] = 0;
else
    sign[j+1] = 1;
if ( sign[j+1] != sign[j] )
{
    it[k1] = t[j-1];
    ft[k1] = t[j];
    k1++;
}
}

/* To find local MLE's of Theta */
for (i=0; i<k1; i++)
{
    c = (ft[i] - it[i])/10.0;
    do
    {
        t[0] = it[i];
        j = 0;
        while( t[j] < ft[i])
        {
            j++;
            t[j] = t[j-1] + c;
        }
        tn = j;
        t[tn] = ft[i];
        for (j=0; j<=tn; j++)
        {
            w = 0.0;
            z = 0.0;
            u = 0.0;
            v = 0.0;
            for (l=0; l<n; l++)
            {
                if (x[l] >= t[j])
                {
                    w = w + fabs(x[l] - t[j]);
                    u = u + pow((x[l] - t[j]), 2.0);
                }
                else
                {
                    z = z + fabs(x[l] - t[j]);
                    v = v + pow((x[l] - t[j]), 2.0);
                }
            }
            rt[j] = pow((w/z), (1.0/2));
            st[j] = pow((u/v), (1.0/3));
            p = (st[j-1]-rt[j-1])*(st[j]-rt[j]);
            if ( j!= 0)
            {
                if ( p < 0 )
                {
                    it[i] = t[j-1];
                    ft[i] = t[j];
                    break;
                }
            }
        }
    }
}

```

```

    }
  }
  c = c/10.0;
}
while (fabs(st[j]-rt[j]) > 0.01);
ltheta[i] = t[j];
la[i] = (1.0 + pow((u/v), (1.0/3)))/2.0;
lb[i] = la[i]/(2.0*la[i] - 1.0);
sgs[i] = ((la[i]*la[i]*v)+(lb[i]*lb[i]*u))/(float)n;
}

/* To find global MLE of Theta */
for (i=0; i<k1; i++)
{
  p = -1.0/(2.0*sgs[i]);
  c = log(pow((1/sqrt(2*22*sqrt(sgs[i])/7)), n));
  sum1 = 0.0;
  sum2 = 0.0;
  for(l=0; l<n; l++)
  {
    if(x[l] <= ltheta[i])
      sum1 = sum1 + pow((x[l] - ltheta[i]), 2.0);
    else
      sum2 = sum2 + pow((x[l] - ltheta[i]), 2.0);
  }
  L[i] = (double)(c+(p*la[i]*la[i]*sum1)+(p*lb[i]*lb[i]*sum2));
}
temp = L[0];
for (i=0; i<k1; i++)
{
  if (L[i] > temp)
  {
    temp = L[i];
    j = i;
  }
}
printf("\n MLE of T occurred at the position %d\n", j+1);
th1 = ltheta[j];
a1 = la[j];
b1 = lb[j];
s1 = sgs[j];
printf ( "\n MLE's of T, a and Sigma_sq are :-\n" );
printf(" Theta = %f, a = %f, b = %f, S^2 = %f\n", th1, a1, b1, s1);

/* To find MLE's of a & Sigma_sq when Theta is known */
w = 0.0;
z = 0.0;
u = 0.0;
v = 0.0;
for (l=0; l<n; l++)
{
  if (x[l] >= theta)
    u = u + pow((x[l] - theta), 2.0);
  else
    v = v + pow((x[l] - theta), 2.0);
}
a2 = (1.0 + pow(u/v, 1/3.0))/2;
b2 = a2/(2*a2-1);
s2 = (a2*a2*v + b2*b2*u)/(float)n;

```

```

printf ( "\n MLE's of a & Sigma_sq when T is known are :-\n" );
printf ( " a = %f, Sigma_sq = %f\n", a2, s2 );

/* To Find MLE's of T and Sigma_sq when a is known */

p = a*a/(b*b);
c = (x[n-1] -x[0])/10.0;
it3 = x[0] + 0.01;
ft3 = x[n-1] - 0.01;
do
{
    t[0] = it3;
    j = 0;
    while( t[j] < ft3 )
    {
        j++;
        t[j] = t[j-1] + c;
    }
    tn = j;
    t[tn] = ft3;
    for (j=0; j<=tn; j++)
    {
        w = 0.0;
        z = 0.0;
        u = 0.0;
        v = 0.0;
        for (l=0; l<n; l++)
        {
            if (x[l] >= t[j])
            {
                w = w + fabs(x[l] - t[j]);
                u = u + pow((x[l] - t[j]), 2.0);
            }
            else
            {
                z = z + fabs(x[l] - t[j]);
                v = v + pow((x[l] - t[j]), 2.0);
            }
        }
        rt[j] = w/z;
        d = (rt[j]-p)*(rt[j-1]-p);
        if ( j != 0 )
        {
            if ( d <= 0 )
            {
                it3 = t[j-1];
                ft3 = t[j];
                break;
            }
        }
    }
    c = c/10.0;
}
while ( fabs(rt[j]-p) > 0.01);
th3 = t[j];
s3 = (a*a*v + b*b*u)/(float)n;
printf ( "\n MLE's of T & Sigma_sq when a is known are :-\n" );
printf ( " Theta = %f, Sigma_sq = %f\n", th3, s3 );
getch();
}

```

Appendix – 3.2

/* To find MLE's of Theta Sigma and a of the given data in case of
drifted Laplace distribution */

```
#include <stdio.h>
#include <conio.h>
#include <math.h>
#include <stdlib.h>
#include <time.h>

void main()
{
    int m, N=268, t, t1, i, j, k, f[1000], fA[1000];
    float ar, w, z, sum1, sum2, th, a, b, s, p, c, temp;
    float y[1000], x[1000], aA[1000], bA[1000], tA[1000], sA[1000];
    double L[1000];
    FILE *fp;
    clrscr();
    fp=fopen("sybsc.txt", "r");
    for(i=0; i<N; i++)
        fscanf(fp, "%f", &x[i]);
    fclose(fp);

    for (i=0; i<N-1; i++)
    {
        for (j=i+1; j<N; j++)
        {
            if (x[i] > x[j])
            {
                temp = x[i];
                x[i] = x[j];
                x[j] = temp;
            }
        }
    }

    /* To find MLE's of Theta, a & Sigma */

    for(i=1; i<N; i++)
    {
        if(x[i] != x[i-1])
        {
            t=i;
            break;
        }
    }
    for(i=N-1; i>0; i--)
    {
        if(x[i] != x[i-1])
        {
            t1=i-1;
            break;
        }
    }
    m=0;
    for(i=t; i<=t1; i++)
    {
        y[m]=x[i];
        m++;
    }
}
```

```

}
tA[0]=x[0];
aA[0]=N;
bA[0]=(float)1/2;
w=0.0;
for(i=0;i<N;i++)
    w=w+fabs(x[i]-x[0]);
sA[0] = (bA[0]*w)/(float)N;
L[0]=-(bA[0]/sA[0])*w;
k=1;
for(i=1;i<m;i++)
{
    w=0.0;
    z=0.0;
    for(j=0;j<N;j++)
    {
        if (x[j] > y[i])
            w = w + fabs(y[i] - x[j]);
        else
            z = z + fabs(y[i] - x[j]);
    }
    ar= (1.0 + pow((w/z), (1.0/2)))/2.0;
    if(ar > N/(2*(i+1)) && ar < N/(2*i))
    {
        aA[k] = ar;
        bA[k] = aA[k]/(2.0*aA[k] - 1.0);
        tA[k] = x[i];
        sA[k] = ((aA[k]*z)+(bA[k]*w))/(float)N;
        L[k] = (aA[k]*z-bA[k]*w);
        k++;
    }
}
tA[k]=x[N-1];
aA[k]=(float)1/2;
bA[k]=N;
w=0;
for(i=0;i<N;i++)
    w=w+fabs(x[i]-x[N-1]);
sA[k] = (aA[k]*w)/(float)N;
L[k]=-(aA[k]/sA[k])*w;

/* To find global MLE of Theta */

temp = L[0];
for (i=0; i<=k; i++)
{
    if (L[i] > temp)
    {
        temp = L[i];
        j = i;
    }
}
th = tA[j];
a = aA[j];
b = bA[j];
s = sA[j];
printf ( "\n MLE's of T, a and Sigma are :-\n" );
printf(" Theta = %f, a = %f, b = %f, sigma = %f\n", th, a, b, s);

getch();
}

```