

Unit 5

Some Distributions Derived from Compounding and Conditioning

Abstract

Let X be a non-negative integer valued random variable (r.v.) and $Y = \sum_{i=1}^X Z_i$ with $Z_1, Z_2, Z_3, \dots$ i.i.d. r.v.s independent of X . Then, distribution of Y is said to be derived by compounding. The joint distribution of (X, Y) be called as a Compound Bivariate distribution. Further, we develop some distributions of a random variable Y , by specifying the conditional distributions of Y given X and the marginal distribution of X . We examine for Y to be able to identify the parameters. For certain pairs of families of distributions we define 'Change of order' relation and illustrative examples are given.

1. Distributions derived by compounding

Let X be a non-negative integer valued random variable and $Z_1, Z_2, Z_3, \dots$ be i.i.d. r.v.s independent of X . Define,

$$Y = \sum_{i=1}^X Z_i \quad (Y = 0 \text{ if } X = 0). \quad (1)$$

Then (X, Y) be said to have Bivariate Compound distribution. The marginal distribution of Y is said to be a compounded distribution. Some real life examples are

- 1) X : no. of customers arrived to a shop in a day,
- Z_i : Goods purchased (in Rs) by the i^{th} customer,
- Y : Goods sold (in Rs.) in a day

Generating Functions

Let $P_X(s) = E(s^X)$ be the probability generating function (p.g.f.) of a random variable X and $P(s_1, s_2) = E(s_1^X s_2^Y)$ be the p.g.f. of the bivariate random vector (X, Y) .

Consider the P.g.f. of (X, Y) , where Y is as defined in (1).

$$\begin{aligned} P(s_1, s_2) &= E(s_1^X s_2^Y) \\ &= E_X [s_1^X (P_Z(s_2))^X] \\ &= P_X(s_1 P_Z(s_2)) \end{aligned} \quad (2)$$

The p.g.f. of Y is given by

$$P_Y(s) = E(s^Y) = P(1, s) = P_X(P_Z(s_2)) \quad (3)$$

In the following, we consider the case when the families of distributions of X and that of Z are one parameter families. Let X have p.d.f. f_{θ_1} and Z have p.d.f. g_{θ_2} . The p.g.f. of X be denoted by $P_X(s, \theta_1)$.

Definition : The r.v. Y is said to retain identifiability if the marginal distribution of Y has parameter (θ_1, θ_2) .

That is Y does not retain identifiability if and only if $\exists (\theta_1, \theta_2) \neq (\theta_1', \theta_2')$ such that $P_X(P_Z(s_2, \theta_2), \theta_1) = P_X(P_Z(s_2, \theta_2'), \theta_1') \quad \forall s_2$. We note that if the distribution of Y depends on (θ_1, θ_2) through a many-one but not one-to-one function of (θ_1, θ_2) then Y does not retain identifiability. In the following we consider some examples and examine for the retainment of identifiability of Y .

Example 1.1 :-

$X \sim P(\lambda), Z_i \sim B(1, p)$, then

$Y | X = x \sim B(x, p)$

$$f(x, y) = \frac{e^{-\lambda} \lambda^x}{x!} \frac{x!}{y!(x-y)!} p^y q^{x-y}$$

$$\begin{aligned}
 f(y) &= \frac{e^{-\lambda} \left(\frac{p}{q}\right)^y}{y!} \sum_{x=y}^{\infty} \frac{(\lambda q)^x}{(x-y)!} \\
 &= \frac{e^{-\lambda} \left(\frac{p}{q}\right)^y}{y!} \sum_{t=0}^{\infty} \frac{(\lambda q)^y (\lambda q)^t}{t!} \\
 &= \frac{e^{-\lambda p} (\lambda p)^y}{y!}
 \end{aligned}$$

$$Y \sim P(\lambda p).$$

Hence Y does not retain the identifiability.

$$f(x/y) = \frac{e^{-q\lambda} (\lambda q)^{(x-y)}}{(x-y)!} \quad x \geq y$$

$$X - Y \mid Y = y \sim P(\lambda q)$$

Example 1.2 :- Let $X \sim W(p)$, $Z_i \sim W(P)$. Hence $P_X(s) = \frac{p}{q} [(1-sq)^{-1} - 1]$

$$\text{and } P_Z(s) = \frac{P}{Q} [(1-sQ)^{-1} - 1].$$

Y is the number of trials to get the X^{th} success.

$Y \mid X = x \sim \text{NB}(x, P)$; Y : number of trials to get x^{th} success.

$$f(x, y) = q^{x-1} p \binom{y-1}{x-1} Q^{y-x} P^x$$

$$Y \sim W(Pp) \text{ and } X \mid Y = y \sim B(y-1, qP/(1-Pp)) + 1$$

The distribution of Y does not retain the identifiability.

Example 1.3 :- $X \sim$ Waiting time (p), $Z_i \sim \text{Normal}(\theta, 1)$. That is $f_1(x) = q^{x-1}p$,
 $x = 1, 2, \dots$ and $Y/X = x \sim N(\theta x, x)$,

$$f(y/x) = \frac{1}{x\sqrt{2\pi}} \exp \left\{ \frac{-y^2}{2x} + y\theta - \frac{\theta^2 x}{2} \right\}.$$

Hence,

$$f(x, y, \theta, p) = q^{x-1} p \frac{1}{x\sqrt{2\pi}} \exp \left\{ \frac{-y^2}{2x} + y\theta - \frac{\theta^2 x}{2} \right\}.$$

The marginal density of Y is

$$\begin{aligned} f(y) &= \sum_{x=0}^{\infty} \frac{e^{\frac{-1}{2x}(y-\theta x)^2}}{\sqrt{2\pi x}} q^{x-1} p \\ &= \frac{p}{q} \frac{e^{y\theta}}{\sqrt{2\pi}} \sum_{x=0}^{\infty} \frac{e^{\frac{-y^2}{2x} - \frac{\theta^2 x}{2}}}{\sqrt{x}} q^x \\ &= \frac{p}{q} \frac{e^{y\theta}}{\sqrt{2\pi}} \sum_{x=0}^{\infty} \frac{e^{\frac{-y^2}{2x} - \frac{\theta^2 x}{2} + x \log q}}{\sqrt{x}} \\ &= \frac{p}{q} \frac{e^{y\theta}}{\sqrt{2\pi}} \sum_{x=0}^{\infty} \frac{e^{\frac{-y^2}{2x} - \left(\frac{\theta^2}{2} - \log q\right)x}}{\sqrt{x}} \\ &= H(y, \theta, \alpha) h(y, \alpha). \end{aligned}$$

Hence Y retains identifiability.

Remark : Note X may be a continuous r.v. and Z(t) be a stochastic process independent of X. Then define

$$Y = \int_0^X Z(t) dt,$$

For example, if X is duration of facility being made available and Z(s) is rate

of utility per unit time then $Y = \int_0^X u(t) dt$ indicates the total utility.

Models for Y can be developed by considering models for X say exponential and $Z(\cdot)$ may be Poisson process.

2. Distributions derived by conditioning

By using the marginal distribution of X and the conditional distribution of Y given X one can obtain the joint distribution of (X, Y) and the marginal distribution of Y .

Example 2.1 :- Let $X \sim \text{Exponential}$ with mean θ and the conditional distribution of Y given $X = x$ be $P(\lambda x)$. Then

$$\begin{aligned}
 P(s_1, s_2) &= \int_0^\infty \sum_{y=0}^x s_1^x s_2^y \frac{1}{\theta} e^{-x/\theta} \frac{e^{-\lambda x} (\lambda x)^y}{y!} dx \\
 &= \int_0^\infty \frac{s_1^x e^{-x/\theta - \lambda x}}{\theta} \sum_{y=0}^x \frac{(s_2 \lambda x)^y}{y!} dx \\
 &= \int_0^\infty \frac{s_1^x e^{-x(\lambda + \frac{1}{\theta}) + x \lambda s_2}}{\theta} dx \\
 &= \int_0^\infty \left[s_1 e^{-\lambda + \lambda s_2} \right]^x \frac{1}{\theta} e^{-x/\theta} dx \\
 &= \int_0^\infty e^{(\log s_1 - \lambda + \lambda s_2 - 1/\theta)x} \frac{1}{\theta} dx \\
 &= \frac{1}{\theta - \log s_1 + \lambda - \lambda s_2 + 1/\theta} \\
 &= [-\theta \log(s_1) + \lambda \theta - \lambda \theta s_2 + 1]^{-1}
 \end{aligned}$$

The p.g.f. of Y is $P(1, s_2) = [\lambda \theta - \lambda \theta s_2 + 1]^{-1}$

Then, Y has Waiting time distribution.

$$f(y) = \left(\frac{\lambda\theta}{1 + \lambda\theta} \right)^y \left(\frac{1}{1 + \lambda\theta} \right) \quad y = 0, 1, 2, \dots$$

Y does not retain identifiability.

The conditional distribution of X given Y is $G(y+1, \left(\frac{1}{1 + \lambda\theta} \right))$.

This conditional distribution also does not retain identifiability.

Example 2.2 :- Consider a random sample of size n from a drifted distribution obtained from standard normal distribution with drift parameter a . In this case, $P[X < 0] = 1/2a$ (refer to the model discussed in Unit 3). Let R be the number of negative observations. Let U and V be the sums of squares of negative and positive observations respectively, then

$R \sim B(n, 1/2a)$ and given $R = r$ ($0 < r < n$) the variables U and V are independent and their distributions are such that

$$\frac{U}{a} \sim \chi_r^2 \text{ and } \frac{V}{b} \sim \chi_{(n-r)}^2,$$

where $1/b = 2 - 1/a$.

If $r = 0$ then U is degenerate at 0 and if $r = n$ then V is degenerate at 0. The joint and marginal distributions do not have closed form expressions.

3. The change of order relation :

Let F_1 and F_2 be respectively one parameter and two parameter families of distributions corresponding to the distributions of X and that of the conditional distribution of Y given $X = x$. Let F_1^* and F_2^* be those of Y and conditional distribution of X given $Y = y$.

Definition : We say that (F_1^*, F_2^*) is obtained by 'change of order' (CO) of (F_1, F_2) and write it as

$$(F_1, F_2) \xrightarrow{CO} (F_1^*, F_2^*).$$

Remark : Note CO is symmetric relation.

Table indicating the CO for certain families

Ex.	F_1	F_2	F_1^*	F_2^*
1.	$P(\lambda)$	$B(x, p)$	$P(p\lambda)$	$P(p\lambda)+y$
2.	$E(1/\theta)$	$P(\lambda x)$	$W(1/(1+\lambda\theta))$	$G(y+1, 1/(1+\lambda\theta))$
3.	$W(p)$	$NB(x, P)$	$W(Pp)$	$B(y-1, qP/(1-pP)) + 1$
4.	$W(p)$	$N(\theta x, x)$	Mixture	Product/Mixture